MATH 340 A Sensitivity Analysis Example from lectures

The following examples have been sometimes given in lectures and so the fractions are rather unpleasant for testing purposes. Note that each question is imagined to be independent; the changes are not intended to be cumulative.

We wish to consider a trucking firm who can choose to purchase three types of trucks subject to three constraints on capital, space and number of drivers. We use variable x_i to denote the number of trucks of type i to be purchased.

	Truck 1	Truck 2	Truck 3	availability
capital	24	40	46	1200 units \$10K
space	1	1	1	30 trucks
drivers	3	6	6	150 people
net profit	12	20	21	

Now setting x_i = number of trucks of type i to be produced we have the LP:

$$\begin{array}{lllll} \max & 12x_1 & +20x_2 & +21x_3 & & \\ & 24x_1 & +40x_2 & +46x_3 & \leq & 1200 \\ & x_1 & +x_2 & +x_3 & \leq & 30 \\ & 3x_1 & +6x_2 & +6x_3 & \leq & 150 \end{array} \qquad x_1, x_2, x_3 \geq 0$$

We get the final dictionary:

$$\begin{array}{rcrrrr} x_4 & = & 40 & +2x_5 & +6x_2 & +\frac{22}{3}x_6 \\ x_1 & = & 10 & -2x_5 & & +\frac{1}{3}x_6 \\ x_3 & = & 20 & +x_5 & -x_2 & -\frac{1}{3}x_6 \\ z & = & 540 & -3x_5 & -x_2 & -3x_6 \end{array}$$

a) Give B^{-1}, appropriately labelled:

$$B^{-1} = \begin{array}{c} \\ x_4 \\ x_1 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & -2 & -\frac{22}{3} \\ 0 & 2 & -\frac{1}{3} \\ 0 & -1 & \frac{1}{3} \end{pmatrix} \end{array}$$

b) Give the marginal values associated with capital,space and drivers:

capital: 0, space: 3, drivers: 3

i.e. extra capital worth 0 so not helpful, extra space worth 3, extra drivers worth 3. But this is only true on the margin, for large changes, these values are unlikely to be valid (although they provide an upper bound). You'd be willing to pay 3 units for a parking spot and sell a parking spot for 3 units.

c) Give values for c_1, c_2, c_3 so $\{x_4, x_1, x_3\}$ still yields an optimal basis:

In this case we wish $c_N^T - c_B^T B^{-1} A_N \leq \mathbf{0}$:

$$\begin{array}{c} \begin{array}{ccc} x_2 & x_5 & x_6 \end{array} \\ \begin{pmatrix} c_2 & 0 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{ccc} x_4 & x_1 & x_3 \end{array} \\ \begin{pmatrix} 0 & c_1 & c_3 \end{pmatrix} \end{array} \begin{array}{c} \\ x_4 \\ x_1 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & -2 & -\frac{22}{3} \\ 0 & 2 & -\frac{1}{3} \\ 0 & -1 & \frac{1}{3} \end{pmatrix} \end{array} \begin{array}{c} \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_2 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 40 & 0 & 0 \\ 1 & 1 & 0 \\ 6 & 0 & 1 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} x_2 & x_5 & x_6 \end{array} \\ \begin{pmatrix} c_2 - c_3 & c_3 - 2c_1 & \frac{1}{3}c_1 - \frac{1}{3}c_3 \end{pmatrix} \end{array}$$

d) For what range on c_1 is the current basis $\{x_4, x_1, x_3\}$ still optimal?

$$c_N^T - c_B^T B^{-1} A_N = [-1, 21 - 2c_1, \frac{1}{3}c_1 - 7) \leq \mathbf{0}.$$

We deduce that $10\frac{1}{2} \leq c_1 \leq 21$. Note that in this range the optimal solution is $x_1 = 10$ and $x_3 = 20$ and so the profit would be $10c_1 + 420$.

e) What is the optimal solution if $c_1 = 10$? We create the updated dictionary and try pivoting to optimality. We believe we are close to optimal since we have made a small change.

$$\begin{array}{rclllll} x_4 & = & 40 & +2x_5 & +6x_2 & +\frac{22}{3}x_6 \\ x_1 & = & 10 & -2x_5 & & +\frac{1}{3}x_6 \\ x_3 & = & 20 & +x_5 & -x_2 & -\frac{1}{3}x_6 \\ z & = & * & +x_5 & -x_2 & -\frac{11}{3}x_6 \end{array}$$

x_5 enters and x_1 leaves

$$\begin{array}{rclllll} x_4 & = & 50 & & & \\ x_5 & = & 5 & & * & \\ x_3 & = & 25 & & & \\ z & = & * & -\frac{1}{2}x_1 & -x_2 & -\frac{11}{3}x_6 \end{array}$$

We have a new optimal solution $x_3 = 25$. The profit hasn't been worked out but you can since it is $25 \cdot 21 = 525$, a slight reduction.

f) What is the optimal solution if $c_1 = 22$?

$$\begin{array}{rclllll} x_4 & = & 40 & +2x_5 & +6x_2 & +\frac{22}{3}x_6 \\ x_1 & = & 10 & -2x_5 & & +\frac{1}{3}x_6 \\ x_3 & = & 20 & +x_5 & -x_2 & -\frac{1}{3}x_6 \\ z & = & * & -23x_5 & -x_2 & +\frac{1}{3}x_6 \end{array}$$

x_6 enters and x_3 leaves

$$\begin{array}{rclllll} x_4 & = & 480 & & & \\ x_1 & = & 30 & & * & \\ x_6 & = & 60 & & & \\ z & = & * & -22x_5 & -2x_2 & -x_3 \end{array}$$

We have a new optimal solution $x_1 = 30$. The profit hasn't been worked out but you can since it is $30 \cdot 22 = 660$. We have made truck 1 valuable so we now buy lots of it.

g) For what range on c_2 is the current basis optimal. We simply need $c_2 - c_B B^{-1} A_2 \leq 0$. Now $c_B B^{-1} = [0, 3, 3]$. So we find that for $c_2 \leq 21$ the the current basis (and solution) remain optimal.

h) What if we introduce a new truck (we'll use variable x_7) with requirements of 45 capital, 1 parking spot, 5 crew and value 22. Sounds like a winner compared with truck 3. We may imagine adding x_7 and the associated date to our original problem and compute

$$c_7 - c_B B^{-1} A_7 = c_7 - \begin{bmatrix} 0 & 3 & 3 \end{bmatrix} \begin{bmatrix} 45 \\ 1 \\ 5 \end{bmatrix} = 4.$$

Thus we would like to buy truck 7. We compute

$$B^{-1} A_7 = \begin{array}{c} \\ x_4 \\ x_1 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & -2 & -\frac{22}{3} \\ 0 & 2 & -\frac{1}{3} \\ 0 & -1 & \frac{1}{3} \end{pmatrix} \end{array} \begin{bmatrix} 45 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 6\frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

We insert this in our dictionary:

$$\begin{array}{rcrrrrr} x_4 & = & 40 & +2x_5 & +6x_2 & +\frac{22}{3}x_6 & -6\frac{1}{3}x_7 \\ x_1 & = & 10 & -2x_5 & & +\frac{1}{3}x_6 & -\frac{1}{3}x_7 \\ x_3 & = & 20 & +x_5 & -x_2 & -\frac{1}{3}x_6 & -\frac{2}{3}x_7 \\ z & = & 540 & -3x_5 & -x_2 & -3x_6 & +4x_7 \end{array}$$

Here x_7 enters and x_4 leaves but there is more pivoting to be done so I'll ignore this case. Note that since e we will have increased z above 540 then we know that track 7 will be purchased.

i) Describe a profitable truck. It is one in which 3 times the space requirement and 3 times the labour requirement is less than the profit.

j) Determine the value of the new solution when we decrease capital by 50, increase space by 5 and decrease labour by 5. Given our marginal values we expect the profit to change by $\sum_i \Delta b_i \cdot y_i = 0\cdot(-5)+3\cdot 5+3\cdot(-5)=0$. But this prediction works if the current basis remains feasible. Check

$$\begin{array}{cc} & \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{array}{c} x_4 \\ x_1 \\ x_3 \end{array} & \begin{pmatrix} 1 & -2 & -\frac{22}{3} \\ 0 & 2 & -\frac{1}{3} \\ 0 & -1 & \frac{1}{3} \end{pmatrix} \end{array} \begin{bmatrix} 1200-50 \\ 30+5 \\ 150-5 \end{bmatrix} = \begin{bmatrix} 40 \\ 10 \\ 20 \end{bmatrix} + \begin{array}{cc} & \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{array}{c} x_4 \\ x_1 \\ x_3 \end{array} & \begin{pmatrix} 1 & -2 & -\frac{22}{3} \\ 0 & 2 & -\frac{1}{3} \\ 0 & -1 & \frac{1}{3} \end{pmatrix} \end{array} \begin{bmatrix} -50 \\ +5 \\ -5 \end{bmatrix} = \begin{bmatrix} \frac{50}{3} \\ \frac{65}{3} \\ \frac{40}{3} \end{bmatrix}$$

Thus we gently increase trucks of type 1 (to take advantage of increased parking and handle decreased labour) while decreasing type 3.

The changes $\Delta b_1, \Delta b_2, \Delta b_3$ can vary quite a bit and still have $B^{-1}\Delta\mathbf{b} \geq \mathbf{0}$.

k) For what values of Δb_2 is the current basis still optimal?

$$B^{-1}\left(\mathbf{b}+\begin{matrix} 0 \\ \Delta b_2 \\ 0 \end{matrix}\right) \geq \mathbf{0}$$

Thus

$$\begin{bmatrix} 40 \\ 10 \\ 20 \end{bmatrix} + \begin{bmatrix} -2\Delta b_2 \\ 2\Delta b_2 \\ -\Delta b_2 \end{bmatrix} \geq \mathbf{0}$$

from which we deduce

$$-5 \leq \Delta b_2 \leq 20$$

i.e. $25 \leq b_2 \leq 50$. The marginal value of 3 is valid in this interval.

What happens if $B^{-1}(\mathbf{b}+\Delta\mathbf{b}) \not\geq \mathbf{0}$?

ℓ) What is the optimal solution if $\Delta b_1 = -100$ and $\Delta b_2 = 10$?

We compute

$$B^{-1}\left(\mathbf{b}+\begin{matrix} -100 \\ 10 \\ 0 \end{matrix}\right) = \begin{bmatrix} 40 \\ 10 \\ 20 \end{bmatrix} + \begin{array}{cc} & \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{array}{c} x_4 \\ x_1 \\ x_3 \end{array} & \begin{pmatrix} 1 & -2 & -\frac{22}{3} \\ 0 & 2 & -\frac{1}{3} \\ 0 & -1 & \frac{1}{3} \end{pmatrix} \end{array} \begin{bmatrix} -100 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} -80 \\ 30 \\ 10 \end{bmatrix} \not\geq \mathbf{0}$$

We would have the dictionary

$$\begin{array}{rcrrrr} x_4 & = & -80 & +2x_5 & +6x_2 & +\frac{22}{3}x_6 \\ x_1 & = & 30 & -2x_5 & & +\frac{1}{3}x_6 \\ x_3 & = & 10 & +x_5 & -x_2 & -\frac{1}{3}x_6 \\ z & = & 540 & -3x_5 & -x_2 & -3x_6 \end{array}$$

Recall that if $c_N^T - c_B^T B^{-1} A_N \le \mathbf{0}^T$, and so $c_B^T B^{-1}$ is a feasible dual solution of objective function value $c_B^T B^{-1}\mathbf{b}$. If we do not have a primal feasible solution because $B^{-1}(\mathbf{b}+\Delta\mathbf{b}) \not\geq \mathbf{0}$, we still have a dual feasible solution and so we seek to find a dual optimal solution (and so a primal optimal solution) by minimizing the dual objective function while preserving feasibility. Following the Dual Simplex method we have x_4 leave and seek the largest λ so that

$$\begin{array}{ccc} x_5 & x_2 & x_6 \end{array}$$
$\begin{pmatrix} -3 & -1 & -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 & 6 & \frac{22}{3} \end{pmatrix} \le \mathbf{0}^T$. We choose $\lambda = 1/6$ and so x_2 enters resulting in the following dictionary.

$$\begin{array}{rclllll}
x_2 & = & \frac{40}{3} & -\frac{1}{5}x_5 & +\frac{1}{6}x_4 & -\frac{11}{9}x_6 \\
x_1 & = & 30 & -2x_5 & & +\frac{1}{3}x_6 \\
x_3 & = & -\frac{10}{3} & +\frac{4}{3}x_5 & -\frac{1}{6}x_4 & +\frac{8}{9}x_6 \\
z & = & 556\frac{2}{3} & -\frac{8}{3}x_5 & -\frac{1}{6}x_4 & -\frac{16}{9}x_6
\end{array}$$

One more pivot required. We choose x_3 to leave and then seek the largest λ so that

$$\begin{array}{ccc} x_5 & x_4 & x_6 \end{array}$$
$\begin{pmatrix} -\frac{8}{3} & -\frac{1}{6} & -\frac{16}{9} \end{pmatrix} + \lambda \begin{pmatrix} \frac{4}{3} & -\frac{1}{6} & \frac{8}{9} \end{pmatrix} \le \mathbf{0}^T$. There is a tie with $\lambda = 2$ and so either x_5 or x_6 enters but following an extended form of Anstee's rule we choose x_5 to enter. We obtain the new dictionary (or at least the parts we need)

$$\begin{array}{rclll}
x_2 & = & \frac{25}{2} & & \\
x_1 & = & 25 & & * \\
x_5 & = & \frac{5}{2} & & \\
z & = & 550 & -2x_3 & -\frac{1}{2}x_4
\end{array}$$

Thus we get an optimal solution $(25, 25/2, 0, 0, 5/2, 0)$ with $z = 550$ (we make more money as predicted) with new marginal values of $1/2$ for capital, 0 for space, and 0 for labour.

Our tie for the entering variable results in a degeneracy in the dual.

m) What is the optimal solution if we add the constraint $x_2 + x_3 \ge 15$? First note that adding a constraint can only decrease the value of the objective function (it may even make the LP ifeasible). The answer here is easy. Our current solution $x_1 = 10$ and $x_3 = 20$ with $z = 540$ remains optimal.

n) What is the optimal solution if we add the constraint $x_2 + x_3 \ge 22$?

We start by noting that our current optimal solution is not feasible. So we have to do something. We add a new slack variable $x_7 = x_2 + x_3 - 22$. We can't introduce this directly into our dictionary because it contains a basic variable but we can express x_3 in terms of non-basic variables and obtain

$$x_7 = x_2 + x_3 - 22 = x_2 + (20 + x_5 - x_2 - \frac{1}{6}x_6) - 22 = -2 + x_5 - \frac{1}{6}x_6$$

We have the dictionary

$$\begin{array}{rclllll}
x_4 & = & 40 & +2x_5 & +6x_2 & +\frac{22}{3}x_6 \\
x_1 & = & 10 & -2x_5 & & +\frac{1}{3}x_6 \\
x_3 & = & 20 & +x_5 & -x_2 & -\frac{1}{3}x_6 \\
x_7 & = & -2 & +x_5 & & -\frac{1}{6}x_6 \\
z & = & 540 & -3x_5 & -x_2 & -3x_6
\end{array}$$

We follow the dual simplex algorithm, attempting to decrease z value while maintaining dual feasibility by having x_7 leave the basis.

$$\begin{array}{ccc} x_5 & x_2 & x_6 \end{array}$$
$\begin{pmatrix} -3 & -1 & -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 0 & -\frac{1}{6} \end{pmatrix} \le \mathbf{0}^T$. We can take $\lambda = 3$ and choose x_5 as the entering

variable. Our new dictionary is:

$$
\begin{array}{rclllll}
x_4 & = & 44 & & & \\
x_1 & = & 6 & & * & \\
x_3 & = & 22 & & & \\
x_5 & = & 2 & & & \\
z & = & 534 & -3x_7 & -x_2 & -4x_6
\end{array}
$$

We have a new optimal solution $x_1 = 6$ and $x_3 = 22$ (thus we satisfy the constraint $x_2 + x_3 \geq 22$) with $z = 534$ (a reduction in profitability from the added constraint). The marginal values are 0 fro capital, 0 for space, 4 for labour and 3 for the new constraint. We assume the new constraint is written as $-x_2 - x_3 \leq -22$ and so increasing -22 to -21 results in less of a restriction and so an expectation of increased profit of 3 units. When LINDO is given the constraint explicitly as $x_2 + x_3 \geq 22$ it will return a dual price of -3 so that if you increase 22 to 23 then you expect profit to drop by 3.

After n questions perhaps you don't need anymore but we discussed in class two questions:

o) How do we delete a variable? Two suggestions were offered. If for example we wished to remove x_3 we could set $c_3 = -1$, making x_3 unprofitable and our sensitivity techniques would drive it to 0. We could add a constraint $x_3 = 0$ (or $x_3 \leq 0$) and proceed as above. If we wished to remove x_2, since it is non-basic we could just delete it from further consideration.

p) How do we delete a constraint? If for example we wished to remove $x_1 + x_2 + x_3 \leq 30$ we could simply change the right hand side to some large number (say 1000) essentially eliminating the constraint. Or we could add a variable to the constraint $x_1 + x_2 + x_3 - x_7 \leq 30$ where $c_7 = 0$. If the constraint is currently non-binding, such as capital, then essentially the constraint has been eliminated.

q) How do we change an entry in A? This is more difficult but still possible. For the column of a nonbasic variable this is reasonable (try it!). In a test environment, only one pivot suffices to get you to optimality but this is unrealistic and for some changes it may be advisable to start from scratch.