Proof of Strong Duality.

Richard Anstee

The following is not the Strong Duality Theorem since it assumes $\mathbf{x}^*$ and $\mathbf{y}^*$ are both optimal.

Theorem Let $\mathbf{x}^*$ be an optimal solution to the primal and $\mathbf{y}^*$ to the dual where

$$\text{primal}\quad \begin{array}{rc} \max & \mathbf{c}\cdot\mathbf{x} \\ & A\mathbf{x}\le\mathbf{b} \\ & \mathbf{x}\ge\mathbf{0} \end{array} \qquad \text{dual}\quad \begin{array}{rc} \min & \mathbf{b}\cdot\mathbf{y} \\ & A^T\mathbf{y}\ge\mathbf{c} \\ & \mathbf{y}\ge\mathbf{0} \end{array}.$$

Then $\mathbf{c}\cdot\mathbf{x}^* = \mathbf{b}\cdot\mathbf{y}^*$.

Proof: Let A be an $m\times n$ matrix. We obtain the Revised Simplex Formulas (our dictionaries!) by first writing $A\mathbf{x}\le\mathbf{b}$ as $[AI]\begin{bmatrix}\mathbf{x}\\ \mathbf{x}_S\end{bmatrix} = \mathbf{b}$ where we have n original variables (typically our decision variables) and m slack variables. Then considering the variables split into the m basic variables and the n non basic variables for some column basis B of $[AI]$ we obtain the Revised Simplex Formulas.

$$\mathbf{x}_B = B^{-1}\mathbf{b} - B^{-1}A_N\mathbf{x}_N$$
$$z = \mathbf{c}_B^TB^{-1}\mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^TB^{-1}A_N)\mathbf{x}_N$$

Assume that the Simplex Method has pivoted to an optimal solution given by a basis B. Thus the current basic feasible solution $\mathbf{x}$ has $\mathbf{c}\cdot\mathbf{x} = \mathbf{c}\cdot\mathbf{x}^*$ and $\mathbf{x}$ is given as $\mathbf{x}_B = B^{-1}\mathbf{b}$ and $\mathbf{x}_N = \mathbf{0}$. The value of the objective function for $\mathbf{x}$ is equal to $\mathbf{c}_B^TB^{-1}\mathbf{b} = \mathbf{c}\cdot\mathbf{x} = \mathbf{c}\cdot\mathbf{x}^*$ since $\mathbf{x}$ and $\mathbf{x}^*$ are asserted to be optimal. The simplex method terminates if the coefficients in the z row are all ≤ 0 and so we assert

$$\mathbf{c}_N^T - \mathbf{c}_B^TB^{-1}A_N \le \mathbf{0}.$$

We can readily assert that

$$\mathbf{c}_B^T - \mathbf{c}_B^TB^{-1}B \le \mathbf{0}$$

since of course $\mathbf{c}_B^T - \mathbf{c}_B^TB^{-1}B = \mathbf{0}$. But now we have a symmetric expression for all variables, namely for all variables x_i

$$c_i - c_B^TB^{-1}A_i \le 0$$

where A_i denotes the column of $[AI]$ indexed by x_i. We regroup the variables into the original variables $\mathbf{x}$ and the slack variables $\mathbf{x}_S$ to obtain

$$\mathbf{c}^T - \mathbf{c}_B^TB^{-1}A \le 0$$

and

$$\mathbf{0}^T - \mathbf{c}_B^TB^{-1}I \le 0$$

If we let

$$\mathbf{c}_B^TB^{-1} = \mathbf{y}^T,$$

we obtain

$$\mathbf{c}^T - \mathbf{c}_B^TB^{-1}A \le 0 \text{ implies } \mathbf{c}^T \le \mathbf{y}^TA \text{ implies } A^T\mathbf{y}\ge\mathbf{c}$$

and

$$\mathbf{0}^T - \mathbf{c}_B^TB^{-1}I \le 0 \text{ implies } \mathbf{0}^T \le \mathbf{y}^T \text{ implies } \mathbf{y}\ge\mathbf{0}.$$

This means that $\mathbf{y}$ is a feasible solution to the dual. Also we compute

$$\mathbf{b}\cdot\mathbf{y} = \mathbf{y}^T\mathbf{b} = \mathbf{c}_B^TB^{-1}\mathbf{b} = \mathbf{c}\cdot\mathbf{x}$$

and so by Weak Duality $\mathbf{x}$ (as given above) is optimal to the primal and $\mathbf{y}$ is optimal to the dual.■

The book has essentially this same standard proof but the proof above has the virtue of showing the power of the matrix notation. Students will use the Revised Simplex formulas in the Revised Simplex method and become familiar with it both in plugging in numbers and also as a matrix formula. We use it below in the result concerning marginal values.

It is useful to apply this theorem with real numbers. In particular, in any final dictionary (yielding an optimal solution) you may read off the optimal dual solution by reading the negatives of the coefficients of the slack variables in the z row. This is what we called the 'magic' coefficients in an earlier lecture. That the dual solution or 'magic' coefficients behave as promised can be shown by a theorem of the alternative.

The full version of the Strong Duality Theorem is the following: If either

i) the primal has an optimal solution or the dual has an optimal solution

or

ii) there exists feasible solutions to both the primal and the dual

then there exists an optimal solution $\mathbf{x}^*$ to the primal and an optimal solution $\mathbf{y}^*$ to the dual with

$$\mathbf{c} \cdot \mathbf{x}^* = \mathbf{b} \cdot \mathbf{y}^*.$$

Proof: If an optimal solution $\mathbf{x}^*$ to the primal is found by the simplex method then our proof above yields that there is an optimal solution x^* to the primal and an optimal solution $\mathbf{y}^*$.

If the dual has an optimal solution $\mathbf{y}^*$ then we can consider it as the primal and, using the fact that the dual of the dual is the primal we can apply our previous proof.

If there exists feasible solutions to both the primal and the dual then we know the primal is not infeasible and also is bounded (by Weak Duality by $\mathbf{b} \cdot \mathbf{y}$) and so by the Fundamental Theorem of Linear Programming, the primal has an optimal solution $\mathbf{x}^*$ and we proceed as in our previous proof. ■

Recall that for the Fundamental Theorem of Linear Programming to hold, we need to have the Simplex Algorithm terminate which requires an anti-cycling rule such as Bland's Rule. We will prove that Bland's Rule avoids cycling at some point in the course.