Math 340 Midterm Wednesday February 22, 2006

Explain your work. Name LP theorems as you use them.

1.[30pts] Solve the following LP using our two phase method with Anstee's rule. You will need a fake pivot to feasibility and two more pivots in Phase One. In Phase Two you will need one pivot. Is the optimal solution unique? Explain.

$$\begin{array}{lrrrcr} \max & x_1 & +7x_2 & -2x_3 & & \\ & x_1 & +3x_2 & -x_3 & \leq & -2 \\ & -x_1 & +x_2 & -2x_3 & \leq & -3 \\ & & x_2 & & \leq & 2 \end{array} \qquad x_1, x_2, x_3 \geq 0$$

2.[20pts] Consider the following LP

$$\begin{array}{lrrrcr} \max & x_1 & +2x_2 & & & \\ & -2x_1 & -x_2 & +7x_3 & \leq & -2 \\ & x_1 & +x_2 & +x_3 & \leq & 9 \\ & x_1 & +2x_2 & & \leq & 14 \end{array} \qquad x_1, x_2, x_3 \geq 0$$

We are given that $x_1 = 4$, $x_2 = 5$, $x_3 = 0$ is an optimal solution. State the dual LP. Determine an optimal dual solution. Would our primal solution remain optimal if we replaced the first inequality of the primal by $x_1 - x_2 + 5x_3 \leq -1$?

3.[20pts] We are given $A, \mathbf{b}, \mathbf{c}$, current basis and B^{-1}. Determine, using our revised simplex methods (with Anstee's rule), the next entering variable (there is one!) and the next leaving variable (there is one!). Give the 'old' B, the 'new' B and the eta matrix E that updates the old B to the new B.

$$A = \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1 & 7 & 3 & 2 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 & 0 & 1 & 0 \\ 1 & 3 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} b \\ \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \end{array} \qquad B^{-1} = \begin{array}{c} \\ x_5 \\ x_3 \\ x_1 \end{array} \begin{array}{c} \begin{array}{ccc} x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \end{array}$$

$$\mathbf{c}^T = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1 & 7 & 3 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \text{basis } \{x_5, x_3, x_1\}$$

4. a)[5pts] State the Strong Duality Theorem.

b)[5pts] Consider an LP which has an optimal (primal) solution and also a dual optimal solution which happens to have the first dual variable $y_1 = 4$. Give the marginal value interpretation of y_1.

5.[20pts] Let A be an $m \times n$ matrix and let $\mathbf{c}$ an $n \times 1$ vector. Assume that for every $\mathbf{x}$ satisfying $A\mathbf{x} \leq \mathbf{0}$ that $\mathbf{c}^T\mathbf{x} \leq 0$. Prove that there exists a $\mathbf{y}$ satisfying $\mathbf{y} \geq \mathbf{0}$ and $A^T\mathbf{y} = \mathbf{c}$.

(Comment (not a hint): this shows that $\mathbf{c}^T\mathbf{x} \leq 0$ is implied by a positive linear combination of the m inequalities $A\mathbf{x} \leq \mathbf{0}$).