MATH 340 Sample Revised Simplex Computations for Quiz 4.

In each of the following questions you are given $A, \mathbf{b}, \mathbf{c}$, current basis and B^{-1}. Determine, using our revised simplex methods, the next entering variable (if there is one), next leaving variable (if there is one) and the new B^{-1} and the next basic feasible solution.

1.

$$A = \begin{array}{c} \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} b \\ \begin{pmatrix} 8 \\ 7 \\ 3 \end{pmatrix} \end{array} \quad \text{basis: } \{x_3, x_1, x_2\}, \ B^{-1} = \begin{array}{c} \\ x_3 \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$c = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 4 & 5 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Solution:

$$\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1} A_N = \begin{array}{c} \begin{array}{cc} x_4 & x_5 \end{array} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{ccc} x_3 & x_1 & x_2 \end{array} \\ \begin{pmatrix} 0 & 4 & 5 \end{pmatrix} \end{array} \begin{array}{c} \\ x_3 \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{cc} x_4 & x_5 \end{array} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} = [-4\,3]$$

Thus we choose x_5 to enter.

$$B^{-1}\mathbf{b} = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}, \qquad B^{-1}A_5 = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}.$$

Now $x_B = B^{-1}\mathbf{b} - B^{-1}A_5 x_5 \geq 0$ implies $x_5 \leq 1$ and so x_3 leaves the basis.

$$\text{new } B^{-1} = \begin{array}{c} \\ x_5 \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1/3 & -2/3 & 1 \\ 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \end{pmatrix} \end{array} \text{(perform pivot on old } B^{-1} \text{ to take } \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \text{ to } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\text{)}.$$

The next basic feasible solution can be obtained by taking $x_5 = 1$ and then we obtain $x_3 = 0, x_1 = 3, x_2 = 2$ (using $x_B = B^{-1}\mathbf{b} - B^{-1}A_5 \cdot 1$) with $x_4 = 0$.

2.

$$A = \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} b \\ \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \end{array}$$

$$c = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 0 & 4 & 2 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\text{current basis: } \{x_2, x_3, x_4\},\ B^{-1} = \begin{array}{c} \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{ccc} x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1/3 & -1/6 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ -1/3 & 1/6 & 5/6 \end{pmatrix} \end{array}$$

Solution:

$$\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1} A_N =$$

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{ccc} x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 4 & 2 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{ccc} x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1/3 & -1/6 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ -1/3 & 1/6 & 5/6 \end{pmatrix} \end{array} \begin{array}{c} \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{cccc} x_1 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} -2 & -2 & 0 & 0 \end{pmatrix} \end{array}$$

Thus we are at optimality (since all coefficients are negative) with an optimal solution:

$$B^{-1}\mathbf{b} = \begin{pmatrix} 1/3 & -1/6 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ -1/3 & 1/6 & 5/6 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 5/3 \\ 1/3 \end{pmatrix}.$$

3.

$$A = \begin{array}{c} \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & -2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} b \\ \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \end{array} \qquad \text{basis: } \{x_1, x_5, x_6\},\ B^{-1} = \begin{array}{c} \\ x_1 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \end{array}$$

$$c = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 3 & -1 & 5 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Solution:

$$\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1} A_N = \begin{array}{c} \begin{array}{ccc} x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} -1 & 5 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{ccc} x_1 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 3 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ x_1 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} -1 & 2 & 1 \\ -2 & 3 & 0 \\ -1 & 1 & 0 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 2 & -1 & -3 \end{pmatrix} \end{array}$$

Thus we choose x_2 to enter.

$$B^{-1}\mathbf{b} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \qquad B^{-1}A_2 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}.$$

Now $B^{-1}\mathbf{b} - B^{-1}A_2 x_2 \geq 0$ implies no restriction on the value of x_2 and so the LP is unbounded.

Note: Letting $x_2 = t > 0$ we have the solution $x_1 = 2 + t, x_5 = 2 + t, x_6 = 1$ of value $z = 6 + 2t$.