MATH 340 Practice for Quiz # 3 October 7, 2010

These practice problems give the spirit of the questions that can be asked. Obviously question 1 could be altered so that you are given a dual optimal solution. And in question 2, a different error/inconsistency could arise.

1.[14 points] The optimal solution to the linear program:

$$\begin{array}{lrrrl} \text{Maximize} & 10x_1 & +14x_2 & +20x_3 & \\ & 2x_1 & +3x_2 & +4x_3 & \leq 220 \\ & 4x_1 & +2x_2 & -x_3 & \leq 385 \\ & x_1 & & +4x_3 & \leq 160 \end{array} \qquad x_1, x_2, x_3 \geq 0$$

is $x_1 = 60, x_2 = 0, x_3 = 25$. Write down the dual problem. Use this information above to find an optimal solution to the dual (**don't use the simplex algorithm**) explaining your work (name theorems used). Explain how this confirms that the optimal solution to the primal I claimed is in fact an optimal solution.

Solution:

$$\text{Dual:} \quad \begin{array}{lrrrl} \text{Minimize} & 220y_1 & +385y_2 & +160y_3 & \\ & 2y_1 & +4y_2 & +y_3 & \geq 10 \\ & 3y_1 & +2y_2 & & \geq 14 \\ & 4y_1 & -y_2 & +4y_3 & \geq 20 \end{array} \qquad y_1, y_2, y_3 \geq 0$$

Now $x_1 = 60 > 0$ implies $2y_1^* + 4y_2^* + y_3^* = 10$ by Complementary Slackness.
Also $x_3 = 25 > 0$ implies $4y_1^* - y_2^* + 4y_3^* = 20$ by Complementary Slackness.
Also $4x_1 + 2x_2 - x_3 = 215 < 385$ implies $y_2^* = 0$ by Complementary Slackness.

The optimal solution to the dual can be determined by solving three equations in 3 unknowns to obtain

$$y_1^* = 5, y_2^* = 0, y_3^* = 0$$

We check feasibilty of our primal and dual solutions and then, since Complementary Slackness is satisfied, the Theorem of Complementary Slackness shows that the primal (and dual) solution is optimal. An alternate way is to note you have a feasible solution to the primal $(60, 0, 25)$ with objective function value $10 \times 60 + 14 \times 0 + 20 \times 25 = 1100$ and a feasible solution to the dual $(5, 0, 0)$ with objective function value $220 \times 5 + 385 \times 0 + 160 \times 0 = 1100$ and so by Weak Duality, both must be optimal.

Comment: It is somewhat lucky that the three equations determine an optimal dual solution but that is how the question was chosen. The value of the question is in testing your hands on understanding of Complementary Slackness.

2. [6 points] Consider the LP:

$$\begin{array}{lrrrrl} \text{Maximize} & 12x_1 & +20x_2 & +21x_3 & +18x_4 & \\ & 24x_1 & +40x_2 & +46x_3 & +44x_4 & \leq 1200 \\ & x_1 & +x_2 & +x_3 & +x_4 & \leq 30 \\ & 3x_1 & +6x_2 & +6x_3 & +6x_4 & \leq 150 \end{array} \qquad x_1, x_2, x_3, x_4 \geq 0$$

Someone claims the final dictionary has

$$z = 540 - x_2 - 3x_4 - 4x_6 - 3x_7$$

Explain what optimal solution to the dual this implies and explain why there must have been an error in the final row for z.

Solution:
Following our proof of the Strong Duality Theorem, and our comments on the magic coefficients, an optimal solution to the dual has

$$y_i^* = -\text{coefficient of the } i\text{th slack of the primal}$$

and so $y_1^* = 0, y_2^* = 4, y_3^* = 3$. But our supposedly optimal solution to the primal has $z = 540$ and yet $1200y_1^* + 30y_2^* + 150y_3^* = 570$ which violates Strong Duality and so some error must have been made. There are a number of ways in which an error can be made including complementary slackness; youll have to hunt a little.

Comments: This quiz is asking you to be able to use our Duality Theorems with real numbers.