MATH 340 Pivots Preserve the Set of All Solutions Richard Anstee

It is crucial that we know that the pivot process preserves the set of solutions to the original dictionary. We will not concern ourselves with positivity of the variables because if all solutions are preserved then so are the positive solutions. Consider a general dictionary with the indices reordered so that the current basis is $x_{n+1}, x_{n+1}, \ldots, x_{n+m}$:

$$\begin{array}{rcl} x_{n+i} & = & b_i - \sum_{j=1}^{n} a_{ij}x_j \qquad i = 1, 2, \ldots, m \\ z & = & const + \sum_{j=1}^{n} c_j x_j \end{array} \qquad (1)$$

Consider a pivot where x_k enters and x_{n+l} leaves. For this we would normally have $c_k > 0$ and $a_{kl} > 0$ but this is irrelevant for what follows. We must have $a_{kl} \neq 0$ for the pivot to make sense.

$$\begin{array}{rcl} x_k & = & \frac{1}{a_{lk}}(b_l - \sum_{j=1, j\neq k}^{n} a_{lj}x_j - x_{n+l}) \\ x_{n+i} & = & b_i - \sum_{j=1, j\neq k}^{n} a_{ij}x_j - \frac{a_{ik}}{a_{lk}}(b_l - \sum_{j=1, j\neq k}^{n} a_{lj}x_j - x_{n+l}) \;\; i = 1, 2, \ldots, m, \;\; i \neq l \\ z & = & const + \sum_{j=1, j\neq k}^{n} c_j x_j + \frac{c_k}{a_{lk}}(b_l - \sum_{j=1, j\neq k}^{n} a_{lj}x_j - x_{n+l}) \end{array} \qquad (2)$$

We are using the substitution method, namely replacing x_k by its new expression. A solution to (1) yields a solution to (2) since the equations in (2) are derived from the equations in (1) by linear combinations.

$$\mathrm{sol}^{\underline{ns}}(1) \subseteq \mathrm{sol}^{\underline{ns}}(2)$$

We can now choose to pivot with x_{n+l} entering and x_k leaving.

$$\begin{array}{rcl} x_{n+l} & = & b_l - \sum_{j=1}^{n} a_{lj}x_j \\ x_{n+i} & = & b_i - \sum_{j=1, j\neq k}^{n} a_{ij}x_j - \frac{a_{ik}}{a_{lk}}(b_l - \sum_{j=1, j\neq k}^{n} a_{lj}x_j - (b_l - \sum_{j=1}^{n} a_{lj}x_j)) \\ & = & b_i - \sum_{j=1, j\neq k}^{n} a_{ij}x_j - \frac{a_{ik}}{a_{lk}}(a_{lk}x_k) \\ & = & b_i - \sum_{j=1}^{n} a_{ij}x_j \qquad i = 1, 2, \ldots, m, i \neq l \\ z & = & const + \sum_{j=1, j\neq k}^{n} c_j x_j + \frac{c_k}{a_{lk}}(b_l - \sum_{j=1, j\neq k}^{n} a_{lj}x_j - (b_l - \sum_{j=1}^{n} a_{lj}x_j)) \\ & = & const + \sum_{j=1, j\neq k}^{n} c_j x_j + \frac{c_k}{a_{lk}}(a_{lk}x_k) \\ & = & const + \sum_{j=1}^{n} c_j x_j \end{array} \qquad (3)$$

We have obtained a new dictionary (3) that is identical to (1)! But as before

$$\mathrm{sol}^{\underline{ns}}(2) \subseteq \mathrm{sol}^{\underline{ns}}(3)$$

We conclude using $\mathrm{sol}^{\underline{ns}}(1) = \mathrm{sol}^{\underline{ns}}(3)$ that

$$\mathrm{sol}^{\underline{ns}}(1) = \mathrm{sol}^{\underline{ns}}(2)$$

and so the pivot operation preserves the set of solutions.