Math 340 Linear Programming as Pig Farming Richard Anstee

This course studies the topic of *Linear Programming.* a better name for the subject would be *Linear Optimization* since we will be optimizing a linear objective function subject to linear constraints. We will discuss this further in the context of standard forms. The term *programming* uses an older interpretation and is *not* referring to Computer Programming most called coding these days. The term programming referred to writing down tables of values of decision variables describing a program of activities. Dynamic Programming was similarly named.

The only prerequisite for the course in a course in Linear Algebra (MATH 221, MATH 152, or MATH 223) so that you can manipulate matrices and understand matrix inverses. We will be expecting some proofs from you in the course. You can learn by example how to argue in a logical way. We also expect lots of logical thinking! The Theory can be a bit challenging and some questions will enable us to distinguish among the top students. Interestingly, the students often have some difficulties with applications where the approach is not so straightforward. Applications of marginal values are an example that require some thinking rather than an algorithmic approach. We do give related questions on assignments that should help you.

Linear Programming from its creation after World War II was dealing with applications. One of the standard problems was the diet problem; how to feed the troops in wartime at the least expense while providing sufficient nutrition. More as a way to be surprising and catch your attention, a colleague of mine call the subject pig farming! Here the problem is feeding pigs at minimum cost while meeting their nutritional requirements (see below). In fact pig farming is more easily seen as a linear program while a diet problem for humans has to bring in extra constraints of variety, taste etc.

Example Diet Problem / Pig Farming

You wish to feed your pigs at minimum costsubject to certain nutritionall requirements (necessary for good growth of the pigs). There are three types of available foods:

	kg corn	kg silage	kg alfalfa
units carb	.9	.2	.4
units protein	3	8	6
units vitamins	1	2	4
cost ¢/kg	7	6	5

In one day the requirements are for 2 units of carbohydrates, 18 units of protein and 15 units of vitamins. We wish to obtain a cheapest diet for the pigs.

To express this mathematically we need to consider what choices we have and we can express these as *decision variables* for these mathematical quantities we wish to determine.

$$\begin{aligned} c &= \text{kg corn} \\ s &= \text{kg silage} \\ a &= \text{kg alfalfa} \end{aligned}$$

The problem becomes:

minimize	cost
subject to	the nutritional constraints

More specifically for this problems data:

$$\begin{array}{lrrrcr}
\text{minimize} & 7c & +6s & +5a & & \\
\text{subject to} & & & & & \\
& .9c & +.2s & +.4a & \geq & 2 \\
& 3c & +8s & +6a & \geq & 18 \\
& c & +2s & +4a & \geq & 15 \\
& & & & \multicolumn{2}{r}{c, s, a \geq 0}
\end{array}$$

Note that the nutritional constraints are written here as "$\geq$". What does this mean? Is it reasonable? What are the units? Does the linearity makes sense? (Is $c + 2s + 4a$ the number of units of vitamins?). The constraints $c, s, a \geq 0$ are called *positivity constraints*. Why do we assume the decision variables are positive? Does negative food make sense? Most decision variables that you encounter are required to be positive. These positivity constraints are handled more easily than the other constraints with our Linear Progamming techniques. Software such as LINDO assumes all variables are required to be positive unless otherwise specified.

What are the charateristics of a Linear Program (LP for short). We have an *objective function* that is llnear in the variables, namely a linear combination of the variables plus perhaps a constant (we show how to remove constants later). We wish to maximize or minimize the objective function. We have *constraints* imposed on the variables which are a linear combination of the variables either being "$\leq$", "+" or "$\geq$" a constant. These are hyperplane constraints but we don't need the geometry to understand them. In particular our pig farming solutions lie in $\mathbf{R}^3$ with axes c, s, a. Our typical LP's have many, many variables and so are in a high dimensional space.

The special constraints of the form

$$\text{variable } \geq 0$$

are called *positivity constraints* and they are usually grouped at the end as in our LP above for the pig farming problem. Almost all decision variables are naturally seen to be positive. When we refer to the number of constraints of an LP, we typically ignore the positivity constraints.

Linear Programming (LP) has wide applicability. Moreover we can solve very large problems. It is relatively easy to solve problems with 1000's of constraints and variables. Sometimes for large problems you'd like to hire an expert to help you but software is available.

Notes

Cannot have a non linear objective function. A non linear LP is sometimes denoted NLP. If the objective function is the sume of a linear function of the variables with (constant times) products of two variables , then the result is denoted a *quadratic* objective function (QP). The Karush Kuhn Tucker theorem indicates why this is a special case and most software packages allow a quadratic function to be treated as special. QP problems can be solved much more efficiently than general NLP problems.

Cannot restrict the variables to be integers. This type of restriction is fairly common and makes applications sometimes quite difficult. An LP that requires variables to be integers is called Integer Linear Programming (ILP) and if only some variables are required to be integers is sometimes called Mixed Integer Linear Programming (MILP) but that distinction is not so crucial. Note that an ILP is typically much, much more difficult to solve than the corresponding LP.

We cannot impose strict inequalities. When we get to our Duality Theorems you might consider the effect of strict inequalities. There are ways where a strict inequality can be handled using integer variables.

‘Real World’ problems often involve some integer variables and special software is available to assist. Branch and Bound is one such idea. A problem with only 200 integer variables might be beyond current computing capabilities. But if you know more about the problem type, you might be able to solve larger problems.