We have already seen that geometry shows up strongly in linear algebra in the rotation matrix $R(\theta)$. There are further remarkable interactions that are important in many applications. One typically sees some of these applications in multivariable calculus.

First we define the *dot product* of two n-tuples (which we generalize later to an *inner product* of vectors).

$$\mathbf{x}\cdot\mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Theorem 0.1 *Thinking of a vector* $\mathbf{x} \in \mathbf{R}^n$, *we have the length of* $\mathbf{x}$ *as*

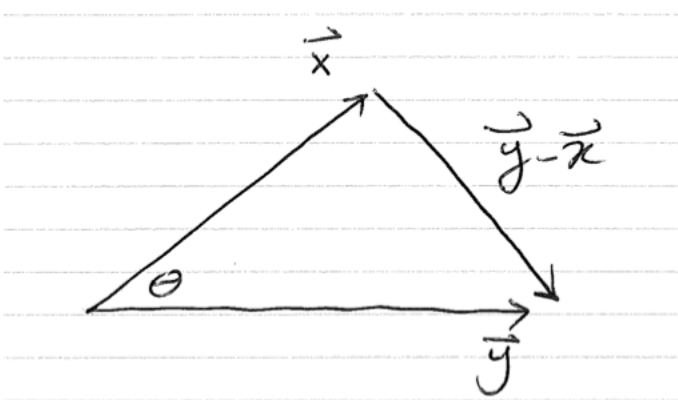
$$||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\mathbf{x}\cdot\mathbf{x}}$$

Proof: Apply induction on n. For $n = 2$, we use the Pythagorean Theorem directly. In general we use $||(x_1, x_2, \ldots, x_{n-1}, 0)^T|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_{m-1}^2}$ (using induction) and $||(0, 0, \ldots, 0, x_n)^T|| = \sqrt{x_n^2}$.Then these vectors are perpendicular (assuming the axes are perpendicular) and lie in a plane (generated by the span of the two vectors) and so we apply the Pythagorean Theorem to obtain the final result. ∎

The dot product has more information.

Theorem 0.2 *If we let* θ *to denote the angle between* $\mathbf{x}$ *and* $\mathbf{y}$ *(in the 2-dimensional plane given as span*$\{\mathbf{x}, \mathbf{y}\}$*), we have* $\mathbf{x}\cdot\mathbf{y} = ||\mathbf{x}||\,||\mathbf{y}||\cos(\theta)$

Proof: We use the Cosine Law on the triangle formed by $\mathbf{x}$,$\mathbf{y}$ and $\mathbf{y}-\mathbf{x}$.



$$||(\mathbf{y}-\mathbf{x})||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2||\mathbf{x}||\,||\mathbf{y}||\cos(\theta)$$

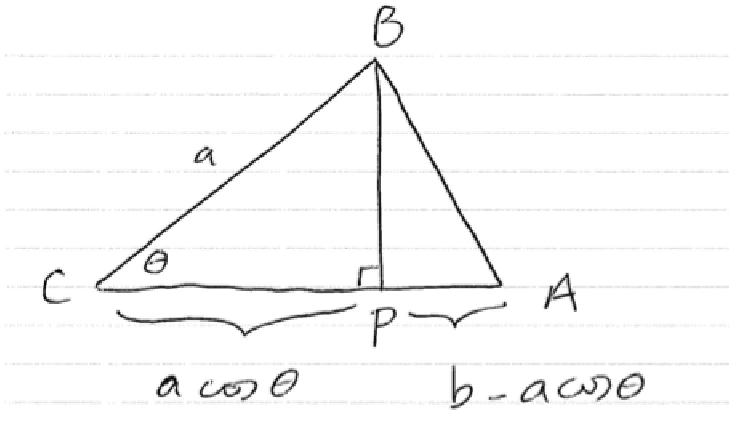
. We have

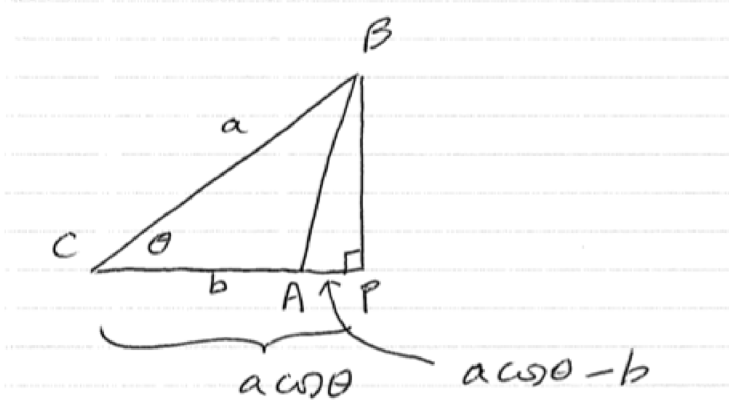
$$||(\mathbf{y}-\mathbf{x})||^2 = (\mathbf{y}-\mathbf{x})\cdot(\mathbf{y}-\mathbf{x}) = \mathbf{y}\cdot\mathbf{y} - \mathbf{y}\cdot\mathbf{x} - \mathbf{x}\cdot\mathbf{y} + \mathbf{x}\cdot\mathbf{x}.$$

We have $\mathbf{x}\cdot\mathbf{x} = ||\mathbf{x}||^2$ and $\mathbf{y}\cdot\mathbf{y} = ||\mathbf{y}||^2$ and $\mathbf{x}\cdot\mathbf{y} = \mathbf{y}\cdot\mathbf{x}$. Matching terms, we obtain our desired result. ∎

We are using the cosine law which is easy enough to verify.

Cosine Law





Proof of Cosine Law: Form a triangle ABC on vertices A, B, C where the side lengths $AB = c$, $AC = b$ and $BC = a$. Let the angle θ be the angle at C. Then drop a perpendicular from B to point P on side AC. Then $BP = a\sin(\theta)$ and $CP = a\cos(\theta)$. We have $AC = b$ and so $PA = b - a\cos(\theta)$. Note this works even if P is not inside the segment AC. Triangle BPA is a right angled triangle and so $c^2 = (a\sin(\theta))^2 + (b - a\cos(\theta))^2 = a^2\sin(\theta)^2 + a^2\cos(\theta)^2 + b^2 - 2ab\cos(\theta)$ yielding $c^2 = a^2 + b^2 - 2ab\cos(\theta)$, the cosine law. Note this works even if P is not inside the segment AC but then using CBP is a rightangled triangle we have $CP = a\cos(\theta)$ and so $AP = a\cos(\theta) - b$ where we note $(a\cos(\theta) - b)^2 = (b - a\cos(\theta))^2$. ■

These lead one to consider what happens when $\mathbf{x} \cdot \mathbf{y} = 0$.

$$\mathbf{x} \text{ and } \mathbf{y} \text{ are orthogonal if } \mathbf{x} \cdot \mathbf{y} = 0$$

This is familiar in 2-dimensional space $\mathbf{R}^2$ and perhaps, depending on your Physics courses, in 3-dimensional space $\mathbf{R}^3$.

It is somehwat arbitrary whether you say $\mathbf{0}$ is orthogonal to another vector. This is reminiscent of dealing with $\mathbf{0}$ when dealing with eigenvectors. As with eigenvectors, we will be looking for a basis for a vector space, in which the vectors are mutually orthogonal, and in that case $\mathbf{0}$ won't appear because it can never be part of a basis.

Interestingly we have orthogonality already appearing in the matrix product $AA^{-1} = I$ where column j of A^{-1} is orthogonal to the ith row of A for $i \neq j$.

Planes

Consider the equation in variables x, y, z

$$ax + by + cz = d$$

We have already identified the solutions as a plane, namely the solutions are

$$\{\mathbf{u} + s\mathbf{v} + t\mathbf{w} \;:\; s, t \in \mathbf{R}\}$$

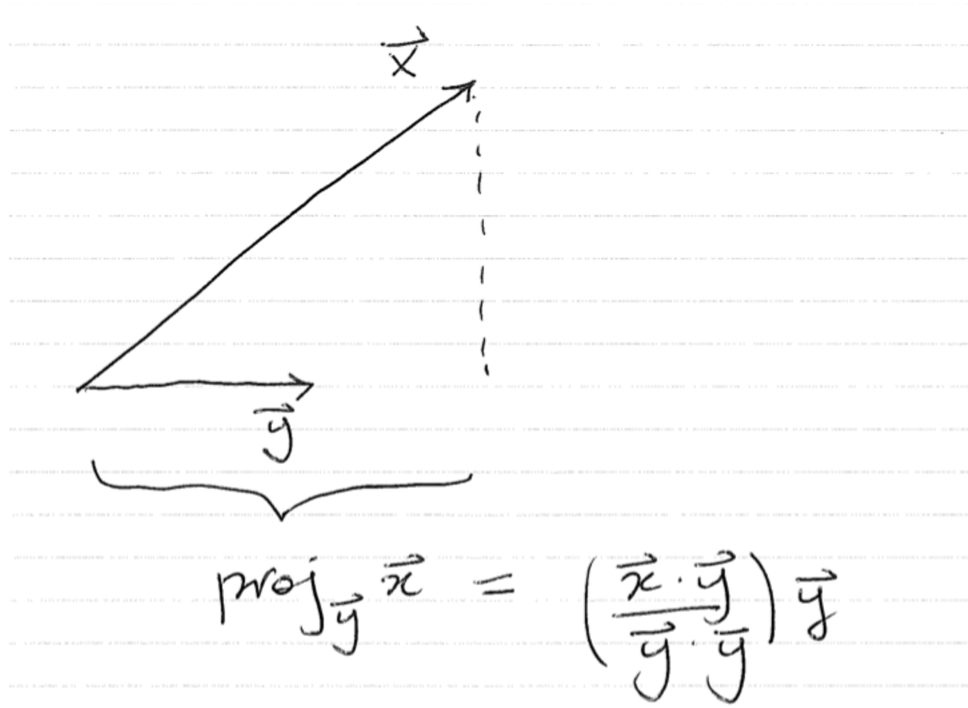
We already know the null space of $ax+by+cz = 0$ is 2-dimensional since the rank of the 1×3 matrix $[a\,b\,c]$ is 1. Let $P = (e, f, g)$ be a point on the plane. Let $\mathbf{p} = (x, y, z)^T$ and $\mathbf{q} = (e, f, g)^T$. Let $\mathbf{n} = (a, b, c)^T$ be called the *normal*. Now $ae + bf + cg = d$ becomes $\mathbf{n}\cdot\mathbf{q} = d$. Also $ax + by + cy = d$ becomes $\mathbf{n}\cdot\mathbf{p} = d$. Then we can rewrite our equation as

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} e \\ f \\ g \end{bmatrix} \right) = \mathbf{n}\cdot(\mathbf{p} - \mathbf{q}) = 0$$

Thinking of a *vector in the plane* as the difference or two points on the plane, we have that any vector in the plane is orthogonal to the normal vector $\mathbf{n}$.

There are many problems to practice here. Find the distance between a pount P and a plane π. Or perhaps the distance between two (non intersecting) planes. Or the distance between two (non intersecting) lines. You would see more of this in a multivariable calculus course where tangent planes are discussed.

Projection



It is possible to obtain a simple formula for the orthogonal *projection* of a vector $\mathbf{x}$ *onto* a vector $\mathbf{y}$, namely a vector $\mathbf{z}$ that is a multiple of $\mathbf{y}$ and so that $\mathbf{x} - \mathbf{z}$ is orthogonal to $\mathbf{y}$. A picture helps with this.

$$\text{proj}_{\mathbf{y}}\mathbf{x} = \frac{\mathbf{x}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}}\mathbf{y}.$$

We check for orthogonality:

$$(\mathbf{x} - \text{proj}_{\mathbf{y}}\mathbf{x})\cdot\mathbf{y} = \mathbf{x}\cdot\mathbf{y} - (\frac{\mathbf{x}\cdot\mathbf{y}}{\mathbf{y}\cdot\mathbf{y}}\mathbf{y})\cdot\mathbf{y} = 0.$$

This yields a wealth of applications. You can compute a number of important geometric quantities such as the distance of a point from a plane (for which orthogonality is seen to be relevant).

Given the equation of a plane $ax + by + cy = d$ we immediately have the normal $\mathbf{n} = (a, b, c)^T$. Given two vectors $\mathbf{u}, \mathbf{v}$ lying in the plane (or 3 points in the plane) we can determine the normal as a non zero vector that is perpendicular to two given vectors $\mathbf{u}, \mathbf{v}$ by solving a system of two equations ($\mathbf{u}\cdot\mathbf{n} = 0$ and $\mathbf{v}\cdot\mathbf{n} = 0$) in 3 unknowns

$$\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Some may wish to compute $\mathbf{n}$ using the *cross product*.

Next we generalize the *dot product* by the *inner product* which has similar properties to the dot product but allows many helpful generalizations including generalizing to two functions.