Math 223 Symmetric and Hermitian Matrices. Richard Anstee

An $n \times n$ matrix Q is *orthogonal* if $Q^T = Q^{-1}$. The columns of Q would form an orthonormal basis for $\mathbf{R}^n$. The rows would also form an orthonormal basis for $\mathbf{R}^n$.

A matrix A is *symmetric* if $A^T = A$.

Theorem 1 *Let A be a symmetric $n \times n$ matrix of real entries. Then there is an orthogonal matrix Q and a diagonal matrix D so that*

$$AQ = QD, \qquad i.e.\ Q^TAQ = D.$$

Note that the entries of Q and D are real.

There are various consequences to this result:

A symmetric matrix A is diagonalizable

A symmetric matrix A has an othonormal basis of eigenvectors.

A symmetric matrix A has real eigenvalues.

Proof: The proof begins with an appeal to the fundamental theorem of algebra applied to $\det(A - \lambda I)$ which asserts that the polynomial factors into linear factors and one of which yields an eigenvalue λ *which may not be real.*

Our second step it to show λ is real. Let $\mathbf{x}$ be an eigenvector for λ so that $A\mathbf{x} = \lambda\mathbf{x}$. Again, if λ is not real we must allow for the possibility that $\mathbf{x}$ is not a real vector.

Let $\mathbf{x}^H = \overline{\mathbf{x}}^T$ denote the conjugate transpose. It also applies to matrices as $A^H = \overline{A}^T$. We will revisit this theorem for Hermitian matrices, namely matrices A with $A^H = A$. Sensibly, Hermitian matrices are allowed to have complex entries.

Now $\mathbf{x}^H\mathbf{x} \geq 0$ with $\mathbf{x}^H\mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$. We compute $\mathbf{x}^HA\mathbf{x} = \mathbf{x}^H(\lambda\mathbf{x}) = \lambda\mathbf{x}^H\mathbf{x}$. Now taking complex conjugates and transpose $(\mathbf{x}^HA\mathbf{x})^H = \mathbf{x}^HA^H\mathbf{x}$ using that $(\mathbf{x}^H)^H = \mathbf{x}$. Then $(\mathbf{x}^HA\mathbf{x})^H = \mathbf{x}^HA\mathbf{x} = \lambda\mathbf{x}^H\mathbf{x}$ using $A^H = A$. Important to use our hypothesis that A is symmetric. But also $(\mathbf{x}^HA\mathbf{x})^H = \overline{\lambda}\mathbf{x}^H\mathbf{x} = \overline{\lambda}\mathbf{x}^H\mathbf{x}$ (using $\mathbf{x}^H\mathbf{x} \in \mathbf{R}$). Knowing that $\mathbf{x}^H\mathbf{x} > 0$ (since $\mathbf{x} \neq \mathbf{0}$) we deduce that $\lambda = \overline{\lambda}$ and so we deduce that $\lambda \in \mathbf{R}$.

The rest of the proof uses induction on n. The result is easy for $n = 1$ ($Q = [1]!$). Assume we have a real eigenvalue λ_1 and a real eigenvector $\mathbf{x}_1$ with $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$ and $||\mathbf{x}_1|| = 1$. We can extend $\mathbf{x}_1$ to an orthonormal basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$. Let $M = [\mathbf{x}_1\,\mathbf{x}_2\,\cdots\mathbf{x}_n]$ be the matrix formed with columns $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. Then

$$AM = M\begin{bmatrix} \lambda_1 & B \\ \mathbf{0} & C \end{bmatrix} \text{ or } M^{-1}AM = \begin{bmatrix} \lambda_1 & B \\ \mathbf{0} & C \end{bmatrix}.$$

which is the sort of result from our assignments. But the matrix on the right is symmetric since it is equal to $M^{-1}AM = M^TAM$ (since the basis was orthonormal) and we note $(M^TAM)^T = M^TAM$ (using $A^T = A$ since A is symmetric). Then B is a $1 \times (n-1)$ zero matrix and C is a symmetric $(n-1) \times (n-1)$ matrix.

By induction there exists an orthogonal matrix N (with $N^T = N^{-1}$) and a diagonal matrix E with $N^{-1}CN = E$. We form a new orthognal matrix

$$P = \begin{bmatrix} 1 & 0\,0 \cdots 0 \\ \mathbf{0} & N \end{bmatrix}$$

which has

$$P^{-1}\left[\begin{array}{cc}\lambda_1 & \mathbf{0}^T \\ \mathbf{0} & C\end{array}\right]P = \left[\begin{array}{cc}\lambda_1 & 0\,0\cdots 0 \\ \mathbf{0} & E\end{array}\right]$$

This becomes

$$P^{-1}M^{-1}AMP = \left[\begin{array}{cc}\lambda_1 & 0\,0\cdots 0 \\ \mathbf{0} & E\end{array}\right]$$

which is a diagonal matrix D. We note that $(MP)^T = P^TM^T = P^{-1}M^{-1}$ and so $Q = MP$ is an orthogonal matrix with $Q^TAQ = D$. This proves the result by induction. ■