Let A be an $m \times n$ matrix. Each column is a vector in $\mathbf{R}^m$ and each row, when interpreted as a column, is a vector in $\mathbf{R}^n$. Let A_i denote the ith column of A. We define the column space of A, denoted colsp(A), as the span$\{A_1, A_2, \ldots, A_n\}$. Similarly we define the row space of A, denoted rowsp(A) as the span of the rows of A, when interpreted as column vectors in $\mathbf{R}^n$.

We have already noted that for $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, we have $A\mathbf{x} = \sum_{i=1}^n x_i A_i \in \text{colsp}(A)$. A consequence is that colsp(A) = Im(f) where we use Im(f) to denote the image space (or range) of the linear transformation $f : \mathbf{R^n} \to \mathbf{R^m}$ given by $f(\mathbf{x}) = A\mathbf{x}$.

We have previously noted the following

Proposition 1 *Let A be an $m \times n$ matrix.*
(a) If M is an $m \times m$ matrix then $\{\mathbf{x} : A\mathbf{x} = \mathbf{0}\} \subseteq \{\mathbf{x} : MA\mathbf{x} = \mathbf{0}\}$
(b) If M is an invertible $m \times m$ matrix, then $\{\mathbf{x} : A\mathbf{x} = \mathbf{0}\} = \{\mathbf{x} : MA\mathbf{x} = \mathbf{0}\}$

We proved (b) at the beginning of the course (in the context of $\{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$ but you can specialize to $\mathbf{b} = \mathbf{0}$). Results related to (a) are often used in midterms.

We can also prove results for rowsp(A) by simply using rowsp(A) = colsp(A^T) but it makes sense to use the staircase pattern obtained by applying Gaussian elimination to A.

Proposition 2 *Let A be an $m \times n$ matrix.*
(a) If M is an $m \times m$ matrix then rowsp(MA) $\subseteq$ rowsp(A)
(b) If M is an invertible $m \times m$ matrix then rowsp(MA) = rowsp(A)

Consider the following example which we imagine was obtained by Gaussian elimination.

$$A = \begin{bmatrix} 2 & -2 & 0 & 2 & 1 & 0 & 0 \\ 4 & -4 & 0 & 4 & 3 & 2 & 2 \\ 2 & -1 & 3 & 4 & 1 & 1 & 2 \\ 2 & 0 & 6 & 6 & 2 & 4 & 8 \end{bmatrix}$$

With E invertible we obtain

$$EA = \begin{bmatrix} 2 & -2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Any linear dependence among the columns such as $y_1A_1 + y_2A_2 + \cdots + y_nA_n = \mathbf{0}$ with $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ yields a solution to $A\mathbf{y} = \mathbf{0}$ and vice versa namely any $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ with $A\mathbf{y} = \mathbf{0}$ yields $y_1A_1 + y_2A_2 + \cdots + y_nA_n = \mathbf{0}$. Let I denote a subset of $\{1, 2, \ldots, n\}$, namely a subset of the column indices. Let A_i denote the ith column of A so that $(EA)_i$ denotes the ith column of EA. We deduce the following using Proposition 1.

Proposition 3 *Let A, E be given with E being invertible. The set of columns $\{A_i : i \in I\}$ is linearly dependent if and only if the set of columns $\{(EA)_i : i \in I\}$ is linearly dependent.*

Corollary 4 *Let A, E be given with E being invertible. It then follows that the set of columns $\{A_i : i \in I\}$ is linearly independent if and only if the set of columns $\{(EA)_i : i \in I\}$ is linearly independent and hence the set of columns $\{A_i : i \in I\}$ forms a basis for* $\text{colsp}(A)$ *if and only if the set of columns $\{(EA)_i : i \in I\}$ forms a basis for* $\text{colsp}(EA)$.

When we look at staircase patterns EA, where E is invertible, it is easy to identify linearly independent columns of EA whose span is $\text{colsp}(EA)$. Given that the sets of columns that are linearly dependent in A are precisely those that are linearly dependent in EA, then it is also true that those that are linearly independent in A are precisely those that are linearly independent in EA. Hence a set of columns of A yielding a column basis for $\text{colsp}(A)$ will correspond to a set of columns of EA yielding a column basis for $\text{colsp}(EA)$. Note that the idea is that the 1st,2nd and 5th columns of EA yield a column basis for $\text{colsp}(EA)$ if and only if the 1st,2nd and 5th columns of A yield a column basis for $\text{colsp}(A)$. It is straightforward to deduce that a basis for $\text{colsp}(EA)$ are columns 1,2 and 5:

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and so, by Corollary 4, a basis for $\text{colsp}(A)$ is

$$\begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}$$

There are other choices for column bases but it is easiest to chose the columns of A whose corresponding columns in EA contain the pivots.

We can now use the (relatively) easy observation that the nonzero rows of EA form a basis for $\text{rowsp}(EA)$. namely a basis for $\text{rowsp}(EA)$ is $\{(2,-2,0,2,1,0,0)^T, (0.1.3.2.0.1.3)^T, (0,0,0,0,1,3,2)^T\}$. Combine this with Proposition 2 with E being invertible and we have that the nonzero rows of EA are also a basis for $\text{rowsp}(A)$.

We have defined $\text{rowsp}(A) = \text{span}\{(2,-2,0,2,1,0,0)^T, (4,-4,0,4,3,2,2)^T, (2,-1,3,4,1,1,3)^T, (2,0,6,6,2,4,8)^T\}$. With E being invertible we have $\text{rowsp}(A) = \text{rowsp}(EA)$ and so a basis for $\text{rowsp}(A)$ is $\{(2,-2,0,2,1,0,0)^T, (0,1,3,2,0,1,3)^T, (0,0,0,0,1,3,2)^T\}$. Please note that E being invertible does not mean that the first 3 rows of A form a basis for $\text{rowsp}(A)$, although it is possible.

Theorem 5 $\dim(\text{rowsp}(A)) = \dim(\text{colsp}(A))$,

Proof: We have $\dim(\text{rowsp}(A))$ being equal to the number of non zero rows of EA and hence the number of pivots and we have $\dim(\text{colsp}(A))$ being equal to the size of a basis for $\text{colsp}(EA)$ which is the number of pivots. ■

Thus Theorem 5 allows us to define

$$\text{rank}(A) = \dim(\text{colsp}(A)) = \dim(\text{rowsp}(A)).$$

From this we obtain the following lovely result. It is often called the Nullity Theorem where nullity is $\dim(\text{nullsp}(A))$.

Theorem 6 *Let A be an $m \times n$ matrix. Then* $\text{rank}(A) + \dim(\text{nullsp}(A)) = n$.

Proof: $\dim(\text{nullsp}(A))$ is the number of free variables. We have the number of pivot variables and the number of free variables is n. ■