Big new concepts in MATH 223 include a *vector space*, *linear independence* (or *linear dependence*), and *dimension*.

Definition *A set* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *of* k *vectors is said to be* linearly dependent *if there are coefficients* $a_1, a_2, \ldots, a_k$ not all zero *such that* $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$.

Definition*A set* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *of* k *vectors is said to be* linearly independent *if when there are coefficients* $a_1, a_2, \ldots, a_k$ *such that* $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$ *then* $a_1 = a_2 = \cdots = a_k = 0$.

Note that $\mathbf{0}$ is a linearly dependent set, since $1 \cdot \mathbf{0} = \mathbf{0}$.

These definitions are more symmetric than for example identifying S as linearly dependent because one vector in S is a linear combination of the others. Note however if $\mathbf{v}_i$ is a linear combination of the other vectors in S, then $\text{span}(S \backslash \mathbf{v}_i) = \text{span}(S)$ and so S is not minimal and S is a linearly independent set of vectors.

Determining Linear Independence for n-tuples is a problem for Gaussian Elimination. Let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ as follows

$$V = \text{span}\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \right\}$$

We might note that $\mathbf{v}_4 = 2\mathbf{v}_1 + \mathbf{v}_3$. Thu s $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. But such clever observations can be discovered by Gaussian Elimination.

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 3 & 5 & 7 \\ 0 & 1 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By elementary row operations we obtain:

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus the set of solutions are

$$\left\{ s \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ -4 \\ 0 \\ 1 \end{bmatrix} \;:\; s, t \in \mathbf{R} \right\}$$

We deduce that $7\mathbf{v}_1 - 4\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$ and $5\mathbf{v}_1 - 4\mathbf{v}_2 + \mathbf{v}_4 = \mathbf{0}$ from which we have $\mathbf{v}_3 = -7\mathbf{v}_1 + 4\mathbf{v}_2$ and $\mathbf{v}_4 = -5\mathbf{v}_1 + 4\mathbf{v}_2$ and so

$$V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$$

Noting that $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent, we have that $\mathbf{v}_1, \mathbf{v}_2$ is a minimal spanning set for V.

Determining whether functions on a domain D are linearly independent is a bit complicated. We can think of a function f as a tuple with number of entries equal to $|D|$. The entries $f(a), f(b), f(c), \ldots$ would be hopeless to list since typically D is infinite (and likely uncountable). But it does indicate a way to show that $f_1, f_2, \ldots, f_k$ are linearly independent by choosing (carefully) k elements $a_1, a_2, \ldots, a_k \in D$ and showing that the k vectors

$$\begin{bmatrix} f_1(a_1) \\ f_1(a_2) \\ \vdots \\ f_1(a_k) \end{bmatrix}, \qquad \begin{bmatrix} f_2(a_1) \\ f_2(a_2) \\ \vdots \\ f_2(a_k) \end{bmatrix}, \qquad \ldots, \begin{bmatrix} f_k(a_1) \\ f_k(a_2) \\ \vdots \\ f_k(a_k) \end{bmatrix}$$

are linearly independent. The reverse of showing the k vectors are linearly dependent does not show that $f_1, f_2, \ldots, f_k$ are linearly dependent but it might suggest a dependency to try out. Verifying a linear dependency for functions involves checking equality for all elements of the Domain and so typically involves using properties of the functions. Note that showing that $\sum_{i=1}^{k} x_i f_i = \mathbf{0}$ for some choice of multipliers $x_1, x_2, \ldots, x_k$, requires showing that $\sum_{i=1}^{k} x_1 f_i(x) = 0$ for *all* $x \in D$. Verifying a linear dependency for functions involves checking equality for all elements of the Domain and so typically involves using properties of the functions.

It makes some sense to choose a minimal subset $S' \subseteq S$ with $\text{span}(S') = \text{span}(S)$. Then S' must be linearly independent. You might note that the span of the empty set is naturally defined to be $\{\mathbf{0}\}$. Such boundary cases can be a bit awkward.

Definition *For a vector space V, a* basis *is a linearly independent set of vectors S so that span$(S) = V$.*

There would be two ways to find a basis. Either begin with a spanning set, and reduce if there are any dependencies. Alternatively build the basis from the ground up as a linearly independent set contained in V