MATH 223. Equiangular lines in d dimensions.

Notes adapted from notes of Aart Blokhuis of Eindhoven. The particular form of the independence proof was suggested by Chris Godsil

Consider the problem of equiangular lines in euclidean space. Lines through the origin in $\mathbf{R}^d$ are one-dimensional subspaces and can be represented by $L =< \mathbf{u} >$ or $L = span(\mathbf{u})$, where $\mathbf{u}$ is one of the two unit vectors on line L. The angle $0 \leq \phi \leq \pi/2$ between two lines $< \mathbf{u} >$ and $< \mathbf{v} >$ is then determined by $\cos\phi = \alpha$, where $\alpha = |(u, v)| = |\mathbf{u}^T\mathbf{v}|$.

A family of lines is called *equiangular* if every pair of them determines the same angle. We are interested in an upper bound for the number of equiangular lines in $\mathbf{R}^d$. This bound is due to Lemmens and Seidel, a polynomial argument is given by Koornwinder, the following "tensor"-argument is essentially equivalent.

So, let $L =< \mathbf{u} >$ be a line in $\mathbf{R}^d$, where vectors are column vectors. We can form the d by d matrix $U = \mathbf{u}\mathbf{u}^T$. This is a symmetric matrix with trace $\text{tr}(U) = 1$ since $\mathbf{u}$ is chosen to be a unit vector. Note U is the projection matrix onto the line $< \mathbf{u} >$.

Now there must be a fixed number $0 < \alpha < 1$ for our collection of lines so for any two different lines given buy unit vectors $< \mathbf{u} >, < \mathbf{v} >$ in our collection we have $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{u} = \alpha$.

Let the equiangular lines in $\mathbf{R}^d$ be given by $\{< \mathbf{v}_i >:\ i = 1, 2, \ldots, t\}$. I claim the associated projection matrices $\{\mathbf{v}_i\mathbf{v}_i^T\ :\ i = 1, 2, \ldots, t\}$ are linearly independent. Assume the contrary, namely there are constants $c_1, c_2, \ldots, c_t$ not all zero so that

$$\sum_{i=1}^{t} c_i \mathbf{v}_i \mathbf{v}_i^T = 0,$$

the righthand side being the $d \times d$ zero matrix. If we multiply on the right by $\mathbf{v}_j$ and on the left by $\mathbf{v}_j^T$ we obtain that

$$\mathbf{v}_j^T \left(\sum_{i=1}^{t} c_i \mathbf{v}_i \mathbf{v}_i^T \right) \mathbf{v}_j = \mathbf{v}_j^T 0 \mathbf{v}_j$$

and so we have (for any choice of j)

$$\left(\sum_{i=1}^{t} c_i \right) \alpha^2 + (1 - \alpha^2) c_j = 0$$

using $\mathbf{v}_j^T\mathbf{v}_i\mathbf{v}_i^T\mathbf{v}_j = \alpha^2$ for $i \neq j$ and $\mathbf{v}_j^T\mathbf{v}_j\mathbf{v}_j^T\mathbf{v}_j = 1$ and with the righthand side being the 1×1 matrix 0. Since $\left(\sum_{i=1}^{t} c_i\right)$ is a constant independent of c_j and $1 - \alpha^2 \neq 0$, then c_j must be the same for each j. Thus $c_1 = c_2 = \cdots = c_t$. But now this yields a contradiction: For example

$$\text{tr}\left(\sum_{i=1}^{t} c_i \mathbf{v}_i \mathbf{v}_i^T \right) = \sum_{i=1}^{t} c_i = tc_1 \quad ,$$

and so $c_1 = c_2 = \cdots = c_t = 0$. This contradiction shows that the t matrices $\mathbf{v}_i\mathbf{v}_i^T$ are linearly independent.

The space of real symmetric $d \times d$ matrices is $d(d + 1)/2$ dimensional (determined by entries below and on the diagonal; you did this kind of idea on an assignment). This gives us the upper bound $d(d + 1)/2$ for the size of a set of equiangular lines in $\mathbf{R}^d$.

It is possible to compute $\alpha = 1/\sqrt{d + 2}$ if the upper bound is realized.

How good is this bound? For $n = 0$ and $n = 1$ we have (formal) equality, but no interesting geometrical picture. For $n = 2$ we get three lines at 60° angles ($\pi/3$) and $\alpha = 1/2$. For $n = 3$ the bound is 6 with $\alpha = 1/\sqrt{5}$. This is realized by the 6 main diagonals of the icosahedron, or to be more explicit: let τ be the positive root of the quadratic $x^2 - x - 1 = 0$, so $\tau = (1 + \sqrt{5})/2$. Now consider the six lines spanned by the 12 vectors $(0, \pm 1, \pm\tau)$, and their cyclic permutations.
The next case of equality is $n = 7$. Here the example (which is unique) is most easily described using the Fano plane, the projective plane of order 2. Consider the 28 lines spanned by the 56 vectors $(\pm 1, \pm 1, 0, \pm 1, 0, 0, 0)$ and their cyclic permutations. Note that the 'support' of each vector corresponds to three collinear points in the Fano plane, a $2-(7,3,1)$ design. Since two lines in the Fano plane intersect in a unique point, all relevant inner products are ± 1.

People can also handle the case $d = 23$.

Generalizations to $\mathbf{C}^d$

Recently much attention has been given to the analogous problem in complex space $\mathbf{C}^d$. A complex line through the origin is again a one dimensional subspace $< \mathbf{u} >$, where we take $\mathbf{u}$ in such a way that $\mathbf{u}^H\mathbf{u} = 1$. $\mathbf{u}^H$ stands for the conjugate transpose of $\mathbf{u}$. To define the 'angle' between two lines we compute $< \mathbf{u}, \mathbf{v} >= \mathbf{u}^H\mathbf{v}$ and take the modulus $\alpha = |\mathbf{u}^H\mathbf{v}|$. As before we can prove an upper bound for the number of equiangular lines. Associate to a line $< \mathbf{u} >$ the d by d matrix $U = \mathbf{u}\mathbf{u}^H$. This is now a hermitian matrix, with trace 1. Hermitian matrices form a *real* vector space of dimension d^2. The dimension follows with two contributions for each off diagonal entry below the main diagonal ($\mathbf{C}$ has dimension 2 over $\mathbf{R}$) and the diagonal entries are all real. As before we get an upper bound, in this case d^2.
Problem: Fill in the details in the above 'proof' and show that in case of equality one has $\alpha = 1/\sqrt{n+1}$.
Problem: Construct a set of 4 equiangular lines in $\mathbf{C}^2$ and 9 in $\mathbf{C}^3$.
Interesting fact: It seems that for all d the bound can be reached, but nobody knows how or why. Sets realizing equality have several interesting names, one of them being SICPOVMS: see for instance:
http://info.phys.unm.edu/papers/reports/sicpovm.html
All known SICPOVMS have been constructed as follows: Consider the subgroup of $Gl(n, \mathbf{C})$ generated by a cyclic permutation of the coordinates, and $\mathrm{diag}(1, \zeta, \ldots, \zeta^{n-1})$ where ζ is a (primitive) n-th root of unity. This group has order n^3 but in its representation on the lines only n^2. For a suitable line $< \mathbf{u} >$ the orbit is a SICPOVM, at least it seems so numerically in dimensions up 45.
Have fun, and let me know if you find anything spectacular, Aart Blokhuis.