MATH 223 Systems of Differential Equations including example with Complex Eigenvalues

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We now consider a system of DE's that has complex eigenvalues. It arises from considering the Differential Equation

$$y'' = -y, \qquad y(0) = 1, y'(0) = 0$$

If we set $y_1(t) = y$ and $y_2(t) = y'$ then we can set

$$\mathbf{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

and then we can write the DE in vector form as

$$\frac{d}{dt}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{y}$$

We can compute eigenvalues and eigenvectors in the natural way using $\mathbf{C}$ instead of $\mathbf{R}$.

$$\underset{A}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} = \underset{M}{\begin{bmatrix} i & -i \\ -1 & -1 \end{bmatrix}} \underset{D}{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}} \underset{M^{-1}}{\begin{bmatrix} -\frac{1}{2}i & -\frac{1}{2} \\ \frac{1}{2}i & -\frac{1}{2} \end{bmatrix}}$$

We could use either of the three methods from above. We can use our third method above (that follows from our change of basis idea). Let $\mathbf{v}_i$ be an eigenvector of eigenvalue λ_i. Then as solution to the DE, ignoring initial conditions, is

$$\mathbf{y} = e^{\lambda_i}\mathbf{v}_i$$

In order to match the initial conditions, we take the appropriate linear combination of these solutions from eigenvector/eigenvalue pairs. In our case we have

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = ae^{it}\begin{bmatrix} i \\ -1 \end{bmatrix} + be^{-it}\begin{bmatrix} -i \\ -1 \end{bmatrix}$$

We can solve for a, b by setting $t = 0$, noting $e^0 = 1$, to obtain

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a\begin{bmatrix} i \\ -1 \end{bmatrix} + b\begin{bmatrix} -i \\ -1 \end{bmatrix} = M\begin{bmatrix} a \\ b \end{bmatrix}$$

We then solve for a, b using M^{-1} to obtain

$$\begin{bmatrix} a \\ b \end{bmatrix} = M^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}i & -\frac{1}{2} \\ \frac{1}{2}i & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}i \\ \frac{1}{2}i \end{bmatrix}$$

We then can compute the solution.

Once, in a previous version of 223, I solved this by substituting

$$e^{it} = \cos(t) + i\sin(t), \quad e^{-it} = \cos(-t) + i\sin(-t) = \cos(t) - i\sin(t)$$

Then I proceeded to solve for a, b which made things much more complicated. Setting $t = 0$ first and then solving for a, b makes things easier. This is easier for computations; both methods spit out an answer. The solution becomes

$$\mathbf{y} = -\frac{1}{2}i(\cos(t) + i\sin(t))\begin{bmatrix} i \\ -1 \end{bmatrix} + \frac{1}{2}i(\cos(t) - i\sin(t))\begin{bmatrix} -i \\ -1 \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

Thus the solution to our DE as expected is $y = \cos(t)$ which has $y(0) = 1$ and $y'(0) = 0$.

We can make some additional simplifications to save work. Let $z = c + di \in \mathbf{C}$. Use the notation $Re(z) = c$ and $Im(z) = d$ to denote the real and imaginary part of z although I would caution that $Im(z) \in \mathbf{R}$. In addition this conflicts with our definition $Im(f)$ referring to the image of the function f. Sigh. We note that $z + \bar{z} \in \mathbf{R}$. Since we are going to get a real solution we can deduce that in the expression

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = a_1 e^{it} \begin{bmatrix} i \\ -1 \end{bmatrix} + a_2 e^{-it} \begin{bmatrix} -i \\ -1 \end{bmatrix}$$

that $\bar{a_1} = a_2$. We can get two different real solutions from the Real and Imaginary parts of one solution

$$e^{it} \begin{bmatrix} i \\ -1 \end{bmatrix} = (\cos t + i \sin t) \begin{bmatrix} i \\ -1 \end{bmatrix} = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} + i \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

$$Re(e^{it} \begin{bmatrix} i \\ -1 \end{bmatrix}) = Re((\cos t + i \sin t) \begin{bmatrix} i \\ -1 \end{bmatrix}) = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}$$

$$Im(e^{it} \begin{bmatrix} i \\ -1 \end{bmatrix}) = Im((\cos t + i \sin t) \begin{bmatrix} i \\ -1 \end{bmatrix}) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

You may verify that the real part comes from the choice $a_1 = 1/2$, $a_2 = 1/2$ and the imaginary part comes from the choice $a_1 = -i/2$, $a_2 = i/2$. We now solve taking a linear combination of these two solutions (which are both real although their origin was complex):

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = a \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} + b \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}, \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We solve and get $a = 0$, $b = 1$ yielding the solution $y_1(t) = \cos t$, $y_2(t) = -\sin t$.

It is not particularly helpful to note that we can compute e^{At} for this A without using complex numbers. For this problem

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

from which we have expresssions for all powers of A. Then

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} -t^2 & 0 \\ 0 & -t^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 0 & -t^3 \\ t^3 & 0 \end{bmatrix} + \frac{1}{4!}\begin{bmatrix} t^4 & 0 \\ 0 & t^4 \end{bmatrix} + \frac{1}{5!}\begin{bmatrix} 0 & t^5 \\ -t^5 & 0 \end{bmatrix} + \cdots$$

$$= \left[\begin{array}{c|c} 1 + 0 - \frac{1}{2!}t^2 + 0 + \frac{1}{4!}t^4 + 0 \cdots & 0 + t + 0 - \frac{1}{3!}t^3 + 0 + \frac{1}{5!}t^5 \cdots \\ \hline 0 + t + 0 - \frac{1}{3!}t^3 + 0 + \frac{1}{5!}t^5 \cdots & 1 + 0 - \frac{1}{2!}t^2 + 0 + \frac{1}{4!}t^4 + 0 \cdots \end{array}\right]$$

$$= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$