Be sure this exam has 3 pages.

THE UNIVERSITY OF BRITISH COLUMBIA

Sessional Examination - December 2008

MATH 223: Linear Algebra

Instructor: Dr. R. Anstee, section 101

Special Instructions: No Aids. No calculators or cellphones. time: 3 hours

You must show your work and explain your answers.

1. [15 marks] Consider the matrix equation $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & -1 & 1 \\ 2 & 3 & 4 & 1 & -1 & 3 \\ 1 & 2 & 2 & 1 & 0 & 2 \\ 1 & 3 & 2 & 2 & 2 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 7 \\ 5 \\ 9 \end{bmatrix}$$

There is an invertible matrix M so that

$$MA = \begin{bmatrix} 1 & 1 & 2 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad M\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

a) [2 marks] What is rank(A)?

b) [4 marks] Give the vector parametric form for the set of solutions to $A\mathbf{x} = \mathbf{b}$.

c) [6 marks] Give a basis for the row space of A. Give a basis for the column space of A. Give a basis for the null space of A.

d) [2 marks] Let A' be the 4×5 matrix obtained by deleting the 5th column of A from A. What is the rank of A'?

2. [15 marks] Let

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Determine an orthonormal basis of eigenvectors and hence an orthogonal matrix Q and a diagonal matrix D so that $A = QDQ^T$. You may find it useful to know that 5 is an eigenvalue of A.

3. [7 marks] Determine the matrix A corresponding to the linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^3$ of projection onto the vector $(1, 2, 3)^T$.

4. [8 marks] Consider the 2×2 matrix A as follows

$$A = \begin{bmatrix} -4 & 4 \\ -12 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}.$$

Define a_n, b_n, c_n, d_n using

$$A^n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$$

Compute

$$\lim_{n\to\infty} \frac{a_n}{b_n}, \qquad \lim_{n\to\infty} \frac{c_n}{d_n}$$

5. [10 marks] You are attempting to solve for x, y, z in the matrix equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Find a 'least squares' choice $\hat{\mathbf{b}}$ in the column space of A (and hence with $||\mathbf{b} - \hat{\mathbf{b}}||^2$ being minimized) and then solve the new system $A\mathbf{x} = \hat{\mathbf{b}}$ for x, y, z.

6. [15 marks] The differentiation operator '$\frac{d}{dx}$' maps (differentiable) functions into functions. The operator can be viewed as a linear transformation on the vector space of differentiable functions. Consider the 3-dimensional vector space P_2 of all polynomials in x of degree at most 2. Then two possible bases for P_2 are $V = \{1, x, x^2\}$ and $U = \{1 + x, x + x^2, x^2 + 1\}$.

   a) [5 marks] Give the 3×3 matrix A representing the linear transformation $\frac{d}{dx}$ acting on P_2 with respect to the basis V.

   b) [5 marks] Give the matrix B representing $\frac{d}{dx}$ with respect to the basis U. You may find it helpful to note that

$$\begin{aligned} 1 &= \tfrac{1}{2}(1+x) - \tfrac{1}{2}(x+x^2) + \tfrac{1}{2}(x^2+1) \\ x &= \tfrac{1}{2}(1+x) + \tfrac{1}{2}(x+x^2) - \tfrac{1}{2}(x^2+1) \\ x^2 &= -\tfrac{1}{2}(1+x) + \tfrac{1}{2}(x+x^2) + \tfrac{1}{2}(x^2+1) \end{aligned}$$

   c) [5 marks] Is matrix A diagonalizable?

7. [10 marks] Let V be a finite dimensional vector space and assume $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is a linearly independent set of k vectors and assume $Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k, \mathbf{y}_{k+1}\}$ is a linearly independent set of $k+1$ vectors. Then show that there is some vector in Y, say $\mathbf{y}_j$, so that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \mathbf{y}_j\}$ is a linearly independent set of $k+1$ vectors.

8. [10 marks] For what values of k is the following matrix diagonalizable?

$$A = \begin{bmatrix} 2 & 0 & 0 \\ k & 1 & 1 \\ 2 & -2 & 4 \end{bmatrix}$$

Hint: determine eigenvalues for A. What is required to make A diagonalizable?

9. [10 marks]
   a) [4 marks] Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal basis for $\mathbf{R}^3$. For any $\mathbf{v} \in \mathbf{R}^3$, if $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_2\mathbf{v}_3$ then show that $\mathbf{v}^T\mathbf{v} = ||\mathbf{v}||^2 = c_1^2 + c_2^2 + c_3^2$.
   b) [6 marks] Let A be a symmetric 3×3 matrix with eigenvalues $\lambda_1 > \lambda_2 > \lambda_3$. Show that

$$\lambda_1 = \max_{\mathbf{x}} \ \mathbf{x}^T A \mathbf{x}$$

   where the maximum is taken over all vectors $\mathbf{x} \in \mathbf{R}^3$ with $\mathbf{x}^T\mathbf{x} = 1$.

100 Total marks