# Forbidden Families of Configurations

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Joint work with Christina Koch
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# Introduction

I have worked with a number of coauthors in this area: Farzin Barekat, Laura Dunwoody, Ron Ferguson, Balin Fleming, Zoltan Füredi, Jerry Griggs, Nima Kamoosi, Steven Karp, Peter Keevash, Christina Koch, Connor Meehan, U.S.R. Murty, Miguel Raggi and Attila Sali but there are works of other authors (some much older, some recent) impinging on this problem as well. For example, the definition of $VC$-dimension uses a forbidden configuration. A survey article is now available at the Electronic Journal of Combinatorics, Dynamic Survey 20.

Christina Koch

Consider the following family of subsets of $\{1,2,3,4\}$:

$$\mathcal{A} = \{\emptyset, \{1,2,4\}, \{1,4\}, \{1,2\}, \{1,2,3\}, \{1,3\}\}$$

The incidence matrix $A$ of the family $\mathcal{A}$ of subsets of $\{1,2,3,4\}$ is:

$$A = \begin{bmatrix} 0 & 1 & \boxed{1} & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

**Definition** We say that a matrix $A$ is *simple* if it is a (0,1)-matrix with no repeated columns.

**Definition** We define $\|A\|$ to be the number of columns in $A$.

$$\|A\| = 6 = |\mathcal{A}|$$

**Definition** Given a matrix $F$, we say that $A$ has $F$ as a *configuration* (denoted $F \prec A$) if there is a submatrix of $A$ which is a row and column permutation of $F$.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \prec \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & 0 & \boxed{0} \end{bmatrix}$$

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**Definitions**

$\mathcal{F} = \{F_1, F_2, \dots, F_t\}$

$\text{Avoid}(m, \mathcal{F}) = \{\, A \,:\, A$ $m$-rowed simple, $F \nprec A$ for all $F \in \mathcal{F}\}$

$\mathit{forb}(m, \mathcal{F}) = \max_A\{\|A\| : A \in \text{Avoid}(m, \mathcal{F})\}$

# Some Main Results

**Definition** Let $K_k$ denote the $k \times 2^k$ simple matrix of all possible columns on $k$ rows.

**Theorem** (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

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When a matrix $A$ has a copy of $K_k$ on some $k$-set of rows $S$, then we say that $A$ shatters $S$. The results of Vapnik and Chervonenkis were for application in Applied Probability, in *Learning Theory*. One defines $A$ to have VC-dimension $k$ if $k$ is the maximum cardinality of a shattered set in $A$. There are further applications; the last CanaDAM and the last SIAM Conference on Discrete Mathematics had plenary talks containing applications.

# Main Bounds

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**Corollary** *Let $F$ be a $k \times \ell$ simple matrix. Then $forb(m, F) = O(m^{k-1})$.*

**Theorem** (Füredi 83). *Let $F$ be a $k \times \ell$ matrix. Then $forb(m, F) = O(m^k)$.*

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**Problem** Given $F$, can we predict the behaviour of $forb(m, F)$?

Let $C_k$ denote the $k \times k$ vertex-edge incidence matrix of the cycle of length $k$.

$$\text{e.g.}\quad C_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, C_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

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Matrices in Avoid$(m, \{C_3, C_5, C_7, \ldots\})$ are called
Balanced Matrices.
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Matrices in Avoid$(m, \{C_3, C_4, C_5, C_6, \ldots\})$ are called
Totally Balanced Matrices.
**Theorem** $\mathit{forb}(m, \{C_3, C_4, C_5, C_6, \ldots\}) = \mathit{forb}(m, C_3)$

The inequality $forb(m, \{C_3, C_4, C_5, C_6, \ldots\}) \leq forb(m, C_3)$ is quite easy.

**Lemma** If $\mathcal{F}' \subset \mathcal{F}$ then $forb(m, \mathcal{F}) \leq forb(m, \mathcal{F}')$.

Obviously the potential difficulty in obtaining equality is a construction but in my Ph.D. thesis I had shown that any $m \times forb(m, C_3)$ simple matrix is in fact totally balanced. Thus we have
$forb(m, \{C_3, C_4, C_5, C_6, \ldots\}) = forb(m, C_3)$.

# A Product Construction

The building blocks of our product constructions are $I$, $I^c$ and $T$:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Definition** Given an $m_1 \times n_1$ matrix $A$ and a $m_2 \times n_2$ matrix $B$ we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of $A$ on top of a column of $B$. If $A$, $B$ are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \times \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} = \begin{bmatrix}1&1&1&0&0&0&0&0&0\\0&0&0&1&1&1&0&0&0\\0&0&0&0&0&0&1&1&1\\1&1&1&1&1&1&1&1&1\\0&1&1&0&1&1&0&1&1\\0&0&1&0&0&1&0&0&1\end{bmatrix}$$

Given $p$ simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the $p$-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.

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# The Conjecture

**Definition** Let $x(F)$ denote the largest $p$ such that there is a $p$-fold product which does not contain $F$ as a configuration where the $p$-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

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**Conjecture** (A, Sali 05) *forb*$(m, F)$ *is* $\Theta(m^{x(F)})$.

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The conjecture has been verified for $k \times \ell$ $F$ where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $\ell = 2$ (A, Keevash 06) and for $k$-rowed $F$ with bounds $\Theta(m^{k-1})$ or $\Theta(m^k)$ plus other cases.

# Forbidden Families can fail Conjecture

**Definition** $\text{ex}(m, H)$ is the maximum number of edges in a (simple) graph $G$ on $m$ vertices that has no subgraph $H$.

$A \in \text{Avoid}(m, \mathbf{1}_3)$ will be a matrix with up to $m+1$ columns of sum 0 or sum 1 plus columns of sum 2 which can be viewed as the vertex-edge incidence matrix of a graph.
Assume $p = |V(H)|$ and $q = |E(H)|$. Let $I(H)$ denote the $p \times q$ vertex-edge incidence matrix associated with $H$.
**Theorem** $\mathit{forb}(m, \{\mathbf{1}_3, I(H)\}) = m + 1 + \text{ex}(m, H)$.
In this talk $I(C_4) = C_4$. (Also $C_4 = I_2 \times I_2$)

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In this talk $I(C_4) = C_4$. (Also $C_4 = I_2 \times I_2$)
**Theorem** $\mathit{forb}(m, \{\mathbf{1}_3, C_4\}) = m + 1 + \text{ex}(m, C_4)$ which is $\Theta(m^{3/2})$.

**Theorem** $\mathit{forb}(m, \{\mathbf{1}_3, C_6\}) = m + 1 + \text{ex}(m, C_6)$ which is $\Theta(m^{4/3})$.

# Forbidden Families can pass Conjecture

**Theorem** (Balogh and Bollobás) Let $k$ be given. Then there is a constant $c_k$ so that $forb(m, \{I_k, I_k^c, T_k\}) = c_k$.

We note that there is no obvious product construction.

Note that $c_k \geq \binom{2k-2}{k-1}$ by taking all columns of column sum at most $k-1$ that arise from the $k-1$-fold product $T_{k-1} \times T_{k-1} \times \cdots \times T_{k-1}$.

Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_\ell\}$.

**Lemma** Let $\mathcal{F}$ and $\mathcal{G}$ have the property that for every $G_i$, there is some $F_j$ with $F_j \prec G_i$. Then $forb(m, \mathcal{F}) \leq forb(m, \mathcal{G})$.

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**Theorem** Let $\mathcal{F}$ be given. Then either there is a constant $c$ with $forb(m, \mathcal{F}) = c$ or $forb(m, \mathcal{F})$ is $\Omega(m)$.

**Proof:** We start using $\mathcal{G} = \{I_p, I_p^c, T_p\}$ with $p$ suitably large.
Either we have the property that there is some $F_r \prec I_p$, and some $F_s \prec I_p^c$ and some $F_t \prec T_p$ in which case
$forb(m, \mathcal{F}) \le forb(m, \{I_p, I_p^c, T_p\}) = O(1)$
or
without loss of generality we have $F_j \not\prec I_p$ for all $j$ and hence $I_m \in \text{Avoid}(m, \mathcal{F})$ and so $forb(m, \mathcal{F})$ is $\Omega(m)$.

# A pair of Configurations with quadratic bounds

$$\underset{I_3}{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}} \times \underset{I_3^c}{\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}} = \begin{bmatrix}1&1&1&0&0&0&0&0&0\\0&0&0&1&1&1&0&0&0\\0&0&0&0&0&0&1&1&1\\0&1&1&0&1&1&0&1&1\\1&0&1&1&0&1&1&0&1\\1&1&0&1&1&0&1&1&0\end{bmatrix}$$

e.g. $F_2(1,2,2,1) = \begin{bmatrix}0&0&0&1&1&1\\0&1&1&0&0&1\end{bmatrix} \notin I \times I^c$.

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$I_{m/2} \times I^c_{m/2}$ is an $m \times m^2/4$ simple matrix avoiding $F_2(1,2,2,1)$, so $forb(m, F_2(1,2,2,1))$ is $\Omega(m^2)$.

(A, Ferguson, Sali 01 $forb(m, F_2(1,2,2,1)) = \lfloor \frac{m^2}{4} \rfloor + \binom{m}{1} + \binom{m}{0}$)

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e.g. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \notin T \times T$.

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$T_{m/2} \times T_{m/2}$ is an $m \times m^2/4$ simple matrix avoiding $I_3$, so $forb(m, I_3)$ is $\Omega(m^2)$.
$(forb(m, I_3) = \binom{m}{2} + \binom{m}{1} + \binom{m}{0})$

By considering the construction $I \times I^c$ that avoids $F_2(1,2,2,1)$ and the construction $T \times T$ that avoids $I_3$, we note that we have only linear obvious constructions that avoid both $F_2(1,2,2,1)$ and $I_3$. We are led to the following:

**Theorem** $forb(m, \{I_3, F_2(1,2,2,1)\})$ is $\Theta(m)$.

More is true:

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We are unable to extend this to the following although it seems to be true.

**Conjecture** $forb(m, \{t \cdot I_3, F_2(1,t,t,1)\})$ is $\Theta(m)$.

This idea was shown to hold for all pairs of the minimal quadratically bounded configurations.

# Standard Induction

Let $A \in \text{Avoid}(m, \mathcal{F})$. Decompose $A$ as follows by deleting row $r$ and collecting any repeated columns in $C_r$:

$$A = \begin{array}{c} \text{row } r \\ \\ \end{array} \left[ \begin{array}{cccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & & C_r & C_r & & D_r \end{array} \right].$$

Now $[B_r C_r D_r] \in \text{Avoid}(m-1, \mathcal{F})$ and so $\|[B_r C_r D_r]\| \leq forb(m-1, \mathcal{F})$. Also $C_r \in \text{Avoid}(m-1, \mathcal{F}')$ where $\mathcal{F}'$ is the (minimal) set of configurations $F'$ such that there is a configuration $F \in \mathcal{F}$ with $F \prec F' \times [0\ 1]$.
We are ready for induction using
$forb(m, \mathcal{F}) \leq forb(m-1, \mathcal{F}) + forb(m-1, \mathcal{F}')$

# Balogh Bollobás extended

Using our very standard induction one can prove the following.

**Theorem** Let $k$ be given. Then $forb(m, \{2 \cdot I_k, 2 \cdot I_k^c, 2 \cdot T_k\})$ is $\Theta(m)$.

**Proof:** We apply the standard induction noting that $C_r \in \text{Avoid}(m, \{I_k, I_k^c, T_k\}$ and so $\|C_r\|$ is $O(1)$ and so by induction $forb(m, \{2 \cdot I_k, 2 \cdot I_k^c, 2 \cdot T_k\})$ is $\Theta(m)$. We note that $I_m \in \text{Avoid}(m, \{2 \cdot I_k, 2 \cdot I_k^c, 2 \cdot T_k\})$.

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**Conjecture** Let $k, t$ be given. Then $forb(m, \{t \cdot I_k, t \cdot I_k^c, t \cdot T_k\})$ is $\Theta(m)$.

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**Theorem** $forb(m,\{t\cdot I_k, t\cdot {I_k}^c, t\cdot T_k\})$ is $\Theta(m)$ for $k=3,4$.

# Balogh Bollobás extended

**Theorem** (A, Meehan 12) Let $\mathcal{F} = \{F_1, F_2, F_3\}$ be a family of $p$-rowed simple matrices with $p \geq k$
such that columns of $F_1|_{\{1,2,\ldots,k\}}$ are contained in $[\mathbf{0}_k\ I_k]$,
such that columns of $F_2|_{\{1,2,\ldots,k\}}$ are contained in $[\mathbf{1}_k\ I_k^c]$
and such that columns of $F_3|_{\{1,2,\ldots,k\}}$ are contained in $[\mathbf{0}_k\ T_k]$.
Then $forb(m, \mathcal{F})$ is $O(m^{p-k})$.

# An unusual Bound

**Theorem** (A,Koch,Raggi,Sali 12) *forb*$(m, \{T_2 \times T_2, I_2 \times I_2\})$ *is* $\Theta(m^{3/2})$.

$$T_2 \times T_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \; I_2 \times I_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

We showed initially that *forb*$(m, \{T_2 \times T_2, T_2 \times I_2, I_2 \times I_2\})$ is $\Theta(m^{3/2})$ but Christina Koch realized that we ought to be able to drop $T_2 \times I_2$ and we were able to redo the proof (which simplified slightly!).

 

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# Induction

Let $A$ be an $m \times forb(m, \mathcal{F})$ simple matrix with no configuration in $\mathcal{F} = \{T_2 \times T_2, I_2 \times I_2\}$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \begin{array}{c} \text{row } r \\ \\ \end{array} \left[ \begin{array}{cccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & & C_r & C_r & & D_r \end{array} \right].$$

# Induction

Let $A$ be an $m \times forb(m, \mathcal{F})$ simple matrix with no configuration in $\mathcal{F} = \{T_2 \times T_2, I_2 \times I_2\}$. We can select a row $r$ and reorder rows and columns to obtain

$$A = \begin{array}{c} \text{row } r \\ \\ \end{array} \left[ \begin{array}{cccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & & C_r & C_r & & D_r \end{array} \right].$$

To show $\|A\|$ is $O(m^{3/2})$ it would suffice to show $\|C_r\|$ is $O(m^{1/2})$ for some choice of $r$. Our proof shows that assuming $\|C_r\| > 20m^{1/2}$ for all choices $r$ results in a contradiction. In particular, associated with $C_r$ is a set of rows $S(r)$ with $S(r) \geq 5m^{1/2}$. We let $S(r) = \{r_1, r_2, r_3, \ldots\}$. After some work we show that $|S(r_i) \cap S(r_j)| \leq 5$. Then we have
$|S(r_1) \cup S(r_2) \cup S(r_3) \cup \cdots|$
$= |S(r_1)| + |S(r_2) \backslash S(r_1)| + |S(r_3) \backslash (S(r_1) \cup S(r_2))| + \cdots$
$= 5m^{1/2} + (5m^{1/2} - 5) + (5m^{1/2} - 10) + \cdots > m$ ! ! !