Unavoidable Multicoloured Families of Configurations

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Abstract

Balogh and Bollobás [*Combinatorica 25, 2005*] prove that for any k there is a constant $f(k)$ such that any set system with at least $f(k)$ sets reduces to a k-star, an k-costar or an k-chain. They proved $f(k) < (2k)^{2^k}$. Here we improve it to $f(k) < 2^{ck^2}$ for some constant $c > 0$.

This is a special case of the following result on the multi-coloured forbidden configurations at 2 colours. Let r be given. Then there exists a constant c_r so that a matrix with entries drawn from $\{0, 1, \ldots, r-1\}$ with at least $2^{c_r k^2}$ different columns will have a $k \times k$ submatrix that can have its rows and columns permuted so that in the resulting matrix will be either $I_k(a, b)$ or $T_k(a, b)$ (for some $a \neq b \in \{0, 1, \ldots, r-1\}$), where $I_k(a, b)$ is the $k \times k$ matrix with a's on the diagonal and b's else where, $T_k(a, b)$ the $k \times k$ matrix with a's below the diagonal and b's elsewhere. We also extend to considering the bound on the number of distinct columns, given that the number of rows is m, when avoiding a $tk \times k$ matrix obtained by taking any one of the $k \times k$ matrices above and repeating each column t times. We use Ramsey Theory.

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1 Introduction

We define a matrix to be *simple* if it has no repeated columns. A (0,1)-matrix that is simple is the matrix analogue of a set system (or simple hypergraph) thinking of the matrix as the element-set incidence matrix. We generalize to allow more entries in our matrices and define an r*-matrix* be a matrix whose entries are in $\{0, 1, \ldots, r-1\}$. We can think of this as an r-coloured matrix. We examine extremal problems and let $\|A\|$ denote the number of columns in A.

We will use the language of matrices in this paper rather than sets. For two matrices F and A, define F to be a *configuration* in A, and write $F \prec A$, if there is a row and column permutation of F which is a submatrix of A. Let $\mathcal{F}$ denote a finite set of matrices. Let

$$\text{Avoid}(m, r, \mathcal{F}) = \{A \,:\, A \text{ is } m\text{-rowed and simple } r\text{-matrix}, F \nprec A \text{ for } F \in \mathcal{F}\}\,.$$

Our extremal function of interest is

$$\text{forb}(m, r, \mathcal{F}) = \max_{A}\{\|A\| \,:\, A \in \text{Avoid}(m, r, \mathcal{F})\}.$$

We use the simplified notation $\text{Avoid}(m, \mathcal{F})$ and $\text{forb}(m, \mathcal{F})$ for $r = 2$. Many results of forbidden configurations are cases with $r = 2$ (and hence $(0,1)-$matrices) and with $|\mathcal{F}| = 1$. There is a survey [4] on forbidden configurations. There are a number of results for general r including a recent general shattering result [7] which has references to earlier work. This paper explores some forbidden families with constant or linear bounds. We do not require any $F \in \mathcal{F}$ to be simple which is quite different from usual forbidden subhypergraph problems.

A lovely result of Balogh and Bollobás on set systems can be restated in terms of a forbidden family. Let I_ℓ denote the $\ell \times \ell$ identity matrix, I_ℓ^c denote the (0,1)-complement of I_ℓ and let T_ℓ be the $\ell \times \ell$ (upper) triangular matrix with a 1 in position (i, j) if and only if $i \le j$. As a configuration, the $\ell \times \ell$ lower triangular matrix with 1's on the diagonal is the same as T_ℓ. The result [5] show that after a constant number of distinct columns, one cannot avoid all three configurations I_ℓ, I_ℓ^c, T_ℓ.

Theorem 1.1 *[5] For any $\ell \ge 2$, forb$(m, \{I_\ell, I_\ell^c, T_\ell\}) \le (2\ell)^{2^\ell}$.*

A slightly worse bound $\text{forb}(m, \{I_\ell, I_\ell^c, T_\ell\}) \le 2^{2^{2\ell}}$ but with a simpler proof is in [6]. We generalize I_ℓ, I_ℓ^c, T_ℓ into r-matrices as follows. Define the generalized identity matrix $I_\ell(a, b)$ as the $\ell \times \ell$ r-matrix with a's on the diagonal and b's elsewhere. Define the generalized triangular matrix $T_\ell(a, b)$ as the $\ell \times \ell$ r-matrix with a's below the diagonal and b's elsewhere. Now we have $I_\ell = I_\ell(1, 0)$, $I_\ell^c = I_\ell(0, 1)$, and $T_\ell = T_\ell(1, 0)$.

Let

$$\mathcal{T}_\ell(r) = \{I_\ell(a,b) \,:\, a,b \in \{0,1,\cdots,r-1\}, a \neq b\} \cup \{T_\ell(a,b) \,:\, a,b \in \{0,1,\cdots,r-1\}, a \neq b\}\,.$$

Note that $I_\ell = I_\ell(1,0)$, $I_\ell^c = I_\ell(0,1)$, $T_\ell = T_\ell(0,1)$, and $T_\ell^c = T_\ell(1,0)$. In Theorem 1.1, we do not have $T_\ell^c = T_\ell(1,0)$ but we note that $T_{\ell-1}(0,1) \prec T_\ell(1,0) \prec T_{\ell+1}(0,1)$. From the point of view of forbidden configurations in the context of Theorem 1.1, $T_\ell(1,0)$ and $T_\ell(0,1)$ are much the same. In general, we have $T_{\ell-1}(a,b) \prec T_\ell(b,a) \prec T_{\ell+1}(a,b)$.

After removing all $T_\ell(a,b)$ with $a > b$, we define a reduced set of configurations:

$$\mathcal{T}_\ell'(r) = \{I_\ell(a,b) \,:\, a,b \in \{0,1,\cdots,r-1\}, a \neq b\} \cup \{T_\ell(a,b) \,:\, a,b \in \{0,1,\cdots,r-1\}, a < b\}\,.$$

We have $|\mathcal{T}_\ell'(r)| = 3\binom{r}{2}$. In particular, $\mathcal{T}_\ell'(2) = \{I_\ell, I_\ell^c, T_\ell\}$ and

$$\mathrm{forb}(m,r,\mathcal{T}_\ell(r)) \le \mathrm{forb}(m,r,\mathcal{T}_\ell'(r)) \le \mathrm{forb}(m,r,\mathcal{T}_{\ell+1}(r)).$$

As a forbidden family, $\mathcal{T}_\ell'(r)$ behaves very much like $\mathcal{T}_\ell(r)$. We will mainly work on $\mathcal{T}_\ell(r)$.

Theorem 1.2 *Let $r \ge 2$ be given. Then there exists a constant c_r so that for any $\ell \ge 1$,*

$$\mathit{forb}(m,r,\mathcal{T}_\ell(r)) \le 2^{c_r \ell^2}.$$

In particular, for $r = 2$, we have

$$\mathit{forb}(m,\{I_\ell, I_\ell^c, T_\ell\}) \le 2^{c_2(\ell^2+\ell)}$$

with the constant $c_2 \le 6\log_2 6 < 15.51$.

The previous best known upper bound for $\mathrm{forb}(m,\{I_\ell, I_\ell^c, T_\ell\})$ (the case $r = 2$) was doubly exponential in ℓ, so this is a substantial improvement. The current best known lower bound for $\mathrm{forb}(m,\{I_\ell, I_\ell^c, T_\ell\})$ is still $\ell^{c_1\ell}$, and it was conjectured $\mathrm{forb}(m,\{I_\ell, I_\ell^c, T_\ell\}) < \ell^{c_2\ell}$ in [5]. We are not quite there yet. The result gives a concrete value for c_2 but it is not likely to be best possible. Currently, for general $r \ge 2$, $c_r < 30\binom{r}{2}^2\log_2 r$. Theorem 1.1 is also generalized to r-colours.

Theorem 1.1 yields a corollary, as noted in [2], identifying which families yield a constant bound and remarking that all other families yield a linear bound. Our multi-coloured extension of Theorem 1.1 also yields a similarly corollary.

Corollary 1.3 *Let $\mathcal{F} = \{F_1, F_2, \ldots, F_p\}$ and r be given. There are two possibilities. Either $\mathit{forb}(m,r,\mathcal{F})$ is $\Omega(m)$ or there exists an ℓ and a function $f : (i,j) \longrightarrow [p]$ defined on all pairs i,j with $i,j \in \{0,1,\ldots,r-1\}$ and $i \neq j$ so that either $F_{f(i,j)} \prec I_\ell(i,j)$ or $F_{f(i,j)} \prec T_\ell(i,j)$, in which case there is a constant $c_{\ell,r}$ with $\mathit{forb}(m,r,\mathcal{F}) = c_{\ell,r}$.*

Proof: Let F_h be $a_h \times b_h$ and let $d = \max_{h\in[p]}(a_h + b_h)$. Then $F_h \nprec I_d(i,j)$ (respectively $F_h \nprec T_d(i,j)$) implies $F_h \nprec I_m(i,j)$ (respectively $F_h \nprec T_m(i,j)$) for any $m \geq d$. Thus if for some choice $i,j \in \{0,1,\cdots,r-1\}$ with $i \neq j$ we have $F_h \nprec I_\ell(i,j)$ or $F_h \nprec T_\ell(i,j)$ for all $h \in [p]$, then forb$(m,\mathcal{F})$ is $\Omega(m)$ using the construction $I_m(i,j)$ or $T_m(i,j)$. ■

Some further applications of our results and proof ideas are in Section 5.

Our forbidden configurations $I_\ell(a,b)$ and $T_\ell(a,b)$ for $a \neq b$ are simple but it is natural to consider non-simple matrices as forbidden configurations. One natural way to create non-simple matrices is as follows. For $t > 1$, let $t\cdot M = [M|M|\cdots|M]$, the concatenation of t copies of M. For a family $\mathcal{F}$ of matrices, we define $t\cdot\mathcal{F} = \{t\cdot M \colon M \in \mathcal{F}\}$. In [3], we showed that forb$(m,\{t\cdot I_k, t\cdot I_k^c, t\cdot T_k\})$ is $O(m)$. We obtain a sharp bound and extend to r-matrices.

Theorem 1.4 *Let $\ell \geq 2$, $r \geq 2$, and $t \geq 1$ be given. Then there is a constant c with*

$$\mathit{forb}(m,r,t\cdot\mathcal{T}_\ell(r)) \leq 2r(r-1)(t-1)m + c.$$

The proof of the upper bound is in Section 4. For a lower bound consider a choice $a,b \in \{0,1,\ldots,r-1\}$ with $a \neq b$. Consider $T_\ell(a,b)$. The first column has (at least) one b and at least one a and the rest either a or b. The following easy result is useful.

Theorem 1.5 *[1] $\mathit{forb}(m,t\cdot[0]) = \lfloor\frac{tm}{2}\rfloor + 1$.*

Given a pair a,b we construct $M_m(a,b)$ as follows. Choose e from $\{a,b\}$. Form an $m \times (\lfloor\frac{tm}{2}\rfloor + 1)$ matrix $M_m(a,b)$ all of whose entries are a or b with $t\cdot[e] \nprec M_m(a,b)$ using Theorem 1.5. Now consider the concatenation M of the $\binom{r}{2}$ matrices $M_m(a,b)$ over all choices for a,b. Thus M is an $m \times (\binom{r}{2}(\lfloor\frac{tm}{2}\rfloor + 1))$ simple r-matrix. Moreover $t\cdot T_\ell^{a,b,d} \nprec M$ since the first column of $T_\ell(a,b)$ has both a's and b's and so must appear in $M_m(a,b)$. But then since $t\cdot[e] \nprec M_m(a,b)$ for some choice $e \in \{a,b\}$, we deduce $t\cdot T_\ell(a,b) \nprec M$. Thus

$$\text{forb}(m,r,t\cdot\mathcal{T}_\ell(r)) \geq \binom{r}{2}(\lfloor\frac{tm}{2}\rfloor).$$

This lower bound is about a quarter of the upper bound in Theorem 1.4. For $\ell \gg \log_2 t$, a different construction in Section 4 improves this lower bound by a factor of 2.

2 Inductive Decomposition

Let M be an m-rowed matrix. Some notation about repeated columns is needed. For an $m\times 1$ column $\alpha \in \{0,1,\ldots,r-1\}^m$, we define $\mu(\alpha,M)$ as the multiplicity of column α in a matrix M. At certain points it is important to consider matrices of bounded column multiplicity. Define a matrix A to be *s-simple* if every column α of A has $\mu(\alpha,A) \leq s$. Simple matrices are 1-*simple*.

We need induction ideas from [3]. Define

$$\text{Avoid}(m, r, \mathcal{F}, s) = \{A : A \text{ is } m\text{-rowed and } s\text{-simple } r\text{-matrix}, F \nprec A \text{ for } F \in \mathcal{F}\},$$

with the analogous definition for $\text{forb}(m, r, \mathcal{F}, s)$. The induction proceeds with a matrix in $\text{Avoid}(m, r, \mathcal{F}, s)$ but the following observation from [3] generalized to r-matrices shows that the asymptotics of $\text{forb}(m, r, \mathcal{F})$ are the same as that of $\text{forb}(m, r, \mathcal{F}, s)$:

$$\text{forb}(m, r, \mathcal{F}) \leq \text{forb}(m, r, \mathcal{F}, s) \leq s \cdot \text{forb}(m, r, \mathcal{F}). \tag{1}$$

The second inequality follows from taking a matrix $A \in \text{Avoid}(m, r, \mathcal{F}, s)$ and forming the matrix A' where $\mu(\alpha, A') = 1$ if and only if $\mu(\alpha, A) \geq 1$ so that $\|A'\| \leq \|A\| \leq s \cdot \|A'\|$.

Let $A \in \text{Avoid}(m, r, \mathcal{F}, s)$. During the proof of Theorem 1.2, $s = \left(\frac{r}{2}\right)^i$ for some i. Assume $\|A\| = \text{forb}(m, r, \mathcal{F}, s)$. Given a row r we permute rows and columns of A to obtain

$$A = \begin{array}{c} \text{row } r \rightarrow \\ \\ \end{array} \begin{bmatrix} 0\,0 \cdots 0 & 1\,1 \cdots 1 & \cdots & r-1\, r-1 \cdots r-1 \\ G_0 & G_1 & & G_{r-1} \end{bmatrix}. \tag{2}$$

Each G_i is s-simple. Note that typically $[G_0 G_1 \cdots G_{r-1}]$ is not s-simple so we cannot use induction directly on $[G_0 G_1 \cdots G_{r-1}]$. We would like to permute the columns of $[G_0 G_1 \cdots G_{r-1}]$ into the form of $[C_1 C_1 A_1]$, where A_1 is an s-simple matrix and C_1 is a matrix such that for each column α of C_1 the copies of α in the two copies of C_1 comes from different G_i and $\mu(\alpha, [C_1 C_1 A_1]) > s$. This can be done greedily. Initially set $A_1 = [G_0 G_1 \cdots G_{r-1}]$. If A_1 is not s-simple, then there is a column α appear in various G_i. Move α to C_1 and delete two copies of α (in two different G_i) from A_1. When the process stops, we get the matrix A_1 and C_1 as stated. Note A_1 is s-simple. For each column α of C_1, the multiplicity of α satisfies

$$s < \mu(\alpha, [C_1 C_1 A_1]) \leq rs.$$

We inductively apply the same decomposition to the $(m-1)$-rowed s-simple matrix A_1 and get a $(m-2)$-rowed s-simple matrix A_2 and an $(m-2)$-rowed matrix C_2 appearing twice, and so on.

We deduce that

$$\|A\| \leq \left(2 \sum_{i=1}^{m-1} \|C_i\|\right) + rs \tag{3}$$

where rs is the maximum number of columns in any 1-rowed s-simple matrix. Note that each C_i is $\frac{rs}{2}$-simple. Note that it is possible to have $\|C_i\| = 0$ in which case we ignore such cases. The idea is to have A as a root of a tree with the children C_i for each i with $\|C_i\| > 0$.

3 Proof of Theorem 1.2

We denote by $R(k_1, k_2, \ldots, k_n)$ the multicolour Ramsey number for the minimum number p of vertices of K_p so that when the edges are coloured using n colours $\{1, 2, \ldots, n\}$ there will be a monochromatic clique of colour i of size k_i for some $i \in \{1, 2, \ldots, n\}$. It is well-known that the multicolour Ramsey number satisfyies the following inequality:

$$R(k_1, k_2, \ldots, k_n) \leq \binom{\sum_{i=1}^{n}(k_i - 1)}{k_1 - 1, \ldots, k_n - 1} < n^{\sum_{i=1}^{n}(k_i - 1)}.$$

This follows from showing that the Ramsey numbers satisfy the same recurrence as the multinomial coefficients but with smaller base cases. The proof of Theorem 1.2 uses the following $2r^2 - r$ multicolour Ramsey number.

Proposition 3.1

$$\textit{Let} \quad u = R(\underbrace{(r-1)(\ell-1)+1, \ldots, (r-1)(\ell-1)+1}_{r \textit{ copies}}, \underbrace{2\ell, \ldots, 2\ell}_{2r(r-1) \textit{ copies}}). \tag{4}$$

Then a upper bound on u is:

$$u \leq (2r^2 - r)^{r(r-1)(5l-3)} < r^{15r(r-1)l}. \tag{5}$$

We highlight the fact that u is bounded by a single exponential function in ℓ for a fixed r (independent of s and m).

We are now going to describe a tree growing procedure used in the proof. We initially start with some $A \in \text{Avoid}(m, r, \mathcal{T}_\ell(r))$ as the root of the tree. Given a matrix $A \in \text{Avoid}(m', r, \mathcal{T}_\ell(r), s)$ that is a node in our tree, we apply the induction ideas of Section 2 to obtain matrices $C_1, C_2, \ldots C_{m'-1}$ and we set the children of A to be the matrices C_i for those i with $\|C_i\| > 0$. Note that $C_i \in \text{Avoid}(m'', r, \mathcal{T}_\ell(r), \frac{rs}{2})$. Repeat.

Lemma 3.2 *Given $A \in \text{Avoid}(m, r, \mathcal{T}_\ell(r))$, form a tree as described above. Then the depth of the tree is at most $\binom{r}{2}(\ell - 1) + 1$.*

Proof: Suppose there is a chain of depth $\binom{r}{2}(\ell - 1) + 2$ in the tree. Pick any column vector α in the matrix forming the terminal node. At its parent node (or row), α is extended twice with some choices a_i, b_i ($a_i \neq b_i$). We label this edge (of the chosen chain) by the colour $\{a_i, b_i\}$. Since the number of colours (each consisting of a pair from $\{0, 1, \ldots, r-1\}$) is at most $\binom{r}{2}$, there is some pair $\{a, b\}$ occurring at least ℓ times (by Pigeonhole principle). As the result, α can be extended into 2^ℓ columns so that the columns form a submatrix B of A that contains the complete $\ell \times 2^\ell$-configuration using only two colours a and b. In particular, A contains $I_\ell(a, b)$ (as well as $T_\ell(a, b)$). This contradiction completes the proof. ■

Lemma 3.3 *Given $A \in \text{Avoid}(m, r, \mathcal{T}_\ell(r))$, form a tree as described above. Then the maximum branching is at most u with u given in (4).*

Proof: $A \in \text{Avoid}(m, r, \mathcal{T}_\ell(r), s)$ be a node of the tree and determine the children of A as described above. Let $|\{i\colon C_i \neq \emptyset\}| \geq u$. Selecting one column $\mathbf{c}_i$ from each non-empty C_i and deleting some rows if necessary, we get the following submatrix of A.

$$\begin{array}{|cc|cc|cc|c}
a_1 & b_1 & * & * & * & * & \cdots \\ \cline{1-2}
 & & a_2 & b_2 & * & * & \\ \cline{3-4}
 & & & & a_3 & b_3 & \\ \cline{5-6}
 & & & & & & \\
\mathbf{c}_1 & \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_3 & \cdots
\end{array} \tag{6}$$

On the diagonal, we can assume $a_i < b_i$ for each i. This is a $u \times 2u$ matrix. We can view this as a $u \times u$ "square" matrix with each entry is a 1×2 row vector. Note that in this "square" matrix, the entries below the diagonal are special 1×2 row vectors of the form (x, x) while the i-th diagonal entries is (a_i, b_i) satisfying $a_i < b_i$. There is no restriction on the entries above the diagonal.

Now we form a colouring of the complete graph K_u. For each edge $ij \in E(K_u)$ (with $i < j$), set the colour of ij to be the combination of the (i, j) entry and the (j, i) entry. Write the (i, j) entry on the top of (j, i) entry to form a 2×2 matrix, which has the following generic form: $\begin{pmatrix} y_1 & y_2 \\ x & x \end{pmatrix}$.

There are at most r^3 such 2×2 matrices and so r^3 colours on which to apply a multicolour version of the Ramsey's theorem. We can reduce the number of colours to $2r^2 - r$, and obtain a better upper bound, by combining some patterns of 2×2 matrices into one colour class to reduce the total number of colours needed. To be precise, we define the colour classes as

$$\bigcup_a \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} \right\} \cup \bigcup_{a \neq b} \left\{ \begin{pmatrix} b & * \\ a & a \end{pmatrix} \right\} \cup \bigcup_{a \neq b} \left\{ \begin{pmatrix} * & b \\ a & a \end{pmatrix} \right\}$$

Note that the matrix $\begin{pmatrix} b_1 & b_2 \\ a & a \end{pmatrix}$ for $b_1 \neq b_2$ fits two colour classes. When this occurs, we break the tie arbitrarily.

A critical idea here is that we only apply Ramsey's theorem once to get a uniform pattern for both entries below and above the diagonal! By the definition of u as a Ramsey number (4), one of the following cases must happen.

Case 1: There is a number $a \in \{0, 1, \ldots, r-1\}$ such that there is a monochromatic clique of size $(r-1)(\ell-1)+1$ using colour $\begin{pmatrix} a & a \\ a & a \end{pmatrix}$.

Since the diagonals have two colours, we can pick one colour other than a and get a square matrix so that all off-diagonal entries are a's and all diagonal elements are not equal to a. Since this matrix has $(r-1)(\ell-1)+1$ rows, by pigeonhole principle, there is a colour, call it b, appearing at least ℓ times on the diagonal. This gives a submatrix $I_\ell(a, b)$ in A, contradicting $A \in \text{Avoid}(m, r, \mathcal{T}_\ell(r), s)$. This eliminates Case 1.

Case 2: There is a pair $a \neq b \in \{0, 1, \dots, r-1\}$ such that there is a monochromatic clique of size 2ℓ using colour $\begin{pmatrix} b & * \\ a & a \end{pmatrix}$.

By selecting first column from each 1×2 entry, we obtain a $(2\ell \times 2\ell)$-square matrix so that the entries below the diagonal are all a's and the entry above the diagonal are all b's. The diagonal entries are arbitrary. By deleting first column, second row, third column, fourth row, and so on, we get a submatrix $T_\ell(a, b)$ in A, contradicting $A \in \text{Avoid}(m, r, \mathcal{T}_\ell(r), s)$. This eliminates Case 2.

Case 3: There is a pair $a \neq b \in \{0, 1, \dots, r-1\}$ such that there is a monochromatic clique of size 2ℓ using colour $\begin{pmatrix} * & b \\ a & a \end{pmatrix}$.

This is symmetric to Case 2, and thus it can be eliminated in the same way.

Thus such an A with u children does not exist. ■

Proof of Theorem 1.2: We do our tree growing beginning with some $A \in \text{Avoid}(m, r, \mathcal{T}_\ell(r))$. We will be applying (3). Regardless the value of m, at most u summands in the summation above are non-zero by Lemma 3.3. It is sufficient to bound each $\|C_i\|$. Recall that each C_i is $\frac{rs}{2}$-simple when the parent node is s-simple.

For $i = 0, 1, 2, \dots, \binom{r}{2}(\ell - 1) + 1$, let $f(i)$ be the maximum value of $\|C\|$ in the i-th depth node of matrix C in the tree above. By convention $f(0) = \|A\|$. Inequality (3) combined with Lemma 3.3 implies the following recursive formula:

$$f(i) \leq 2uf(i+1) + r\left(\frac{r}{2}\right)^i.$$

By Lemma 3.2, we have the initial condition $f(\binom{r}{2}(\ell-1)+1) \leq r \cdot (\frac{r}{2})^{\binom{r}{2}(\ell-1)+1}$ by (3), where a matrix in a node of the tree at depth $\binom{r}{2}(\ell-1)+1$ is $(\frac{r}{2})^{\binom{r}{2}(\ell-1)+1}$-simple.

Pick a common upper bound, say $r^{15r(r-1)\ell}$, for both $2u$ and $r(\frac{r}{2})^{\binom{r}{2}(\ell-1)+1} + 1$. It implies

$$f(i) + 1 \leq (f(i+1) + 1)r^{15r(r-1)\ell}.$$

Thus

$$\|A\| < f(0) + 1 \leq \left(r^{15r(r-1)\ell}\right)^{\binom{r}{2}(\ell-1)+2} \leq r^{30\binom{r}{2}^2\ell^2}.$$

For the special case $r = 2$, the upper bound can be reduced. First, each diagonal entry of the matrix in Equation (6) is always [0 1]. In the proof Lemma 3.3, Case 2 and Case 3, there is no need to delete rows and columns alternatively. The size of the monochromatic clique can be taken to ℓ instead of 2ℓ. Thus in this setting we may take $u = R(\ell, \ell, \ell, \ell, \ell, \ell) = R_6(\ell)$ and obtain

$$|\{i \colon C_i \neq \emptyset\}| < R_6(\ell).$$

Second, all C_i are simple matrices ($\frac{r}{2} = 1 = s$). So the recursive formula for $f(i)$ is

$$f(i) \leq 2R_6(\ell)f(i+1) + 2$$

with the initial condition $f(\ell) \leq 2$. It is not hard to check that $f(\ell-1) \leq 2\ell-1$ for $\ell \geq 2$. (Otherwise, we get a row vector consisting of ℓ 0's or ℓ 1's, which can be used to extend into $T_\ell(0,1)$ or $T_\ell(1,0)$.) Use the bound $R_6(\ell) < 6^{6(\ell-1)}$ and solve the recursive relation for $f(i)$. We get

$$f(0) < \left(2 \cdot 6^{6(\ell-1)}\right)^{\ell-1} \cdot (2\ell-1) \leq 6^{6(\ell-1)\ell}.$$

Thus,

$$\text{forb}(m, 2, \mathcal{T}_\ell(2)) < 6^{6(\ell-1)\ell}.$$

This implies

$$\text{forb}(m, \{I_\ell, I_\ell^c, T_\ell\}) \leq \text{forb}(m, 2, \mathcal{T}_{\ell+1}(2)) < 6^{6\ell(\ell+1)},$$

yielding the stated bound. ■

4 A non-simple forbidden family

In many examples when computing $\text{forb}(m, t \cdot F)$, the proof ideas for $\text{forb}(m, F)$ are important. A much weaker linear bound than Theorem 1.4 for $r = 2$ is in [3] (the constant multiplying m is the constant of Theorem 1.1). The upper bound of Theorem 1.4 is only off by a factor of 2 from the lower bound asymptotically. In fact, Theorem 1.4 can be generalized to s-simple matrices.

Theorem 4.1 *Let $\ell \geq 2$, $r \geq 2$, $t \geq 1$, and $s \geq t-1$ be given. Then there is a constant $c_{\ell,r,t}$ with*

$$\mathit{forb}(m, r, t \cdot \mathcal{T}_\ell(r), s) \leq 2r(r-1)(t-1)m + c_{\ell,r,t} + rs.$$

This upper bound is only off by a factor of 2 from the lower bound. Note that we can use this bound with $s < t-1$ as noted in (1). This yields Theorem 1.4. The proof appears below.

Theorem 4.2 *Let $\ell \geq 3$, $r \geq 2$, $t \geq 1$, and $s \geq t-1$ be given. Then*

$$\mathit{forb}(m, r, t \cdot \mathcal{T}_\ell(r), s) \geq r(r-1)(t-1)m.$$

Proof: Consider the matrix M_m obtained by the concatenation of all matrices in $(t-1) \cdot \{I_m(a,b)\colon \text{ for } a, b \in \{0, 1, \ldots, r-1\}, a \neq b\}$. The matrix M_m is $(t-1)$-simple and hence s-simple with

$$\|M_m\| = r(r-1)(t-1)m.$$

It suffices to show that $M_m \in \text{Avoid}(m, r, t \cdot \mathcal{T}_\ell(r))$. For a choice $a, b \in \{0, 1, \ldots, r-1\}$ with $a \neq b$, we need show $t \cdot I_\ell(a,b) \nprec M_m$ and $t \cdot T_\ell(a,b) \nprec M_m$.

Suppose not, say $t \cdot I_\ell(a,b) \prec M_m$. There are a list of ℓ-rows $i_1, i_2, \ldots, i_\ell$ and $t\ell$ columns $j_1^1, \cdots, j_r^1, j_1^2, \cdots, j_r^2, \cdots, j_1^t, \cdots, j_r^t$, evidencing the copies of $t \cdot I_\ell(a,b)$ in M_m. Let us restrict to the rows $i_1, i_2, \ldots, i_\ell$ at the moment. For each row i_h, there are t

columns who has a at row i_h and b at other $\ell-1$ rows. These columns can only show up in exactly one column in each copy of $I_m(a,b)$. But we only have $t-1$ copies of $T_m(a,b)$. Thus $t\cdot I_\ell(a,b) \not\prec M_m$.

Similarly suppose $t\cdot T_\ell(a,b) \prec M_m$. With $\ell \geq 3$, then there are t columns of M_m and ℓ rows $i_1, i_2, \ldots, i_\ell$ containing a in row i_1 and containing b's in the other $\ell-1$ rows. Each copy of $I_m(a,b)$ contains exactly one such columns with an a in row i_1 and all other copies of $I_m(c,d)$ do not contain such columns of a's and b's. But there are only $t-1$ copies of $I_m(a,b)$. Thus $t\cdot T_\ell(a,b) \not\prec M_m$. ■

Note that in this construction of the lower bound, every column has equal multiplicity $t-1$. By adding $q := \lceil log_2(t-1)\rceil$ rows we can distinguish $t-1$ columns and obtain a $(m+q)$-rowed simple matrix $M' \in \text{Avoid}(m+q, r, t\cdot\mathcal{T}_{\ell-q})$. Thus we have the following corollary.

Corollary 4.3 *Let $t \geq 1$, $r \geq 2$, and $\ell \geq \lceil \log_2 t\rceil + 3$ be given. There is a constant c with forb$(m, r, t\cdot\mathcal{T}_\ell(r)) \geq r(r-1)(t-1)m - c$.* ■

We begin the proof of Theorem 4.1. Consider $(t-1)$-simple matrices. Consider some $A \in \text{Avoid}(m, r, t\cdot\mathcal{T}_\ell(r))$ so that $A \in \text{Avoid}(m, r, t\cdot\mathcal{T}_\ell(r), t-1)$. Apply our inductive decomposition of Section 2 with the bound (3).

Keep track of the matrices C_i that are generated yielding the following structure in A:

$$
\begin{array}{|c|c|c|c|c|c|c}
a_1^1\, a_2^1 \cdots & b_1^1\, b_2^1 \cdots & * & * & * & * & \cdots \\
\cline{1-2}
 & & a_1^2\, a_2^2 \cdots & b_1^2\, b_2^2 \cdots & * & * & \\
\cline{3-4}
 & & & & a_1^3\, a_2^3 \cdots & b_1^3\, b_2^3 \cdots & \\
\cline{5-6}
 & & & & & & \\
C_1 & C_1 & C_2 & C_2 & C_3 & C_3 & \cdots
\end{array}
\qquad (7)
$$

In what follows let

$$T = r(r-1)(t-1)+1. \qquad (8)$$

By the construction, we may require $a_j^i < b_j^i$ for all $i, j = 1, 2, \ldots, m-1$. In analogy to u of (4), let v be the multicolour Ramsey number:

$$v = R(\underbrace{(r-1)(\ell-1)+1, \ldots, (r-1)(\ell-1)+1}_{r^T \text{ copies}},\ \underbrace{2t\ell, \ldots, 2t\ell}_{2Tr(r-1) \text{ copies}}). \qquad (9)$$

Lemma 4.4 *In the inductive structure of (7), we have $|\{i\colon \|C_i\| \geq T\}| < v$.*

Proof: Assume

$$|\{i : \|C_i\| \geq T\}| \geq v. \qquad (10)$$

In what follows we arrive at a contradiction. A has a structure as in (7). Select rows i for which $\|C_i(a,b)\| \geq T$ to obtain an v-rowed matrix as follows. For a given i, select T columns from $C_i(a,b)$ and we have

$$\begin{bmatrix} a_1^i a_2^i \cdots a_T^i & b_1^i b_2^i \cdots b_T^i \\ \alpha & \alpha \\ \beta & \beta \\ \delta & \delta \\ \vdots & \vdots \end{bmatrix} \text{ is a submatrix of } \begin{bmatrix} a_1^i a_2^i \cdots & b_1^i b_2^i \cdots \\ C_i & C_i \end{bmatrix},$$

where each entry $\alpha, \beta, \ldots$ and $[a_1^i a_2^i \cdots a_T^i]$ and $[b_1^i b_2^i \cdots b_T^i]$ are $1 \times T$ row vectors. Now with v choices i with $\|C_i\| \geq T$ we obtain a $v \times 2Tv$ r-matrix X as follows:

$$\left[\begin{array}{cc|cc|cc|cc|c} a_1^{i_1} \cdots a_T^{i_1} & b_1^{i_1} \cdots b_T^{i_1} & * & * & * & * & * & * & \cdots \\ \alpha & \alpha & a_1^{i_2} \cdots a_T^{i_2} & b_1^{i_2} \cdots b_T^{i_2} & * & * & * & * & \cdots \\ \beta & \beta & \gamma & \gamma & a_1^{i_3} \cdots a_T^{i_3} & b_1^{i_3} \cdots b_T^{i_3} & * & * & \cdots \\ \delta & \delta & \mu & \mu & \nu & \nu & a_1^{i_4} \cdots a_T^{i_4} & b_1^{i_4} \cdots b_T^{i_4} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \end{array}\right],$$

where the entries $\alpha, \beta, \ldots$ are $1 \times T$ row vectors with entries from $\{0, 1, \ldots, r-1\}$.

View this matrix as a $v \times v$ "square" matrix with each entry being a $1 \times 2T$ row vector. All entries below the diagonal are "doubled" row vectors, i.e., the concatenation of two identical $1 \times T$ row vectors. All diagonal entries are the concatenation of two $1 \times T$ row vectors, where each coordinate of the first vector is always strictly less than the corresponding coordinate of the second vector. There is no restriction on the entries above the diagonal.

Now we form a colouring of the complete graph K_v. For each edge $ij \in E(K_v)$ (with $i < j$), colour ij using the combination of the (i,j) entry and the (j,i) entry. Write the (i,j) entry on the top of (j,i) entry to form a $2 \times 2T$ matrix, which has the following generic form: $\begin{pmatrix} \beta_1 & \beta_2 \\ \alpha & \alpha \end{pmatrix}$. Here α, β_1, β_2 are $1 \times T$ row vectors.

There are at most r^{3T} such matrices. Instead of applying Ramsey's theorem with r^{3T} colours, we can reduce the total number of colours needed by combining some patterns of $2 \times 2T$ matrices into a single colour class.

The first type of colour classes is denoted $C(\alpha)$ (with $\alpha \in \{0, 1, \ldots, r-1\}^T$) and consists of patterns $\begin{pmatrix} \alpha & \alpha \\ \alpha & \alpha \end{pmatrix}$. The second type of colour classes is denoted $C(a, b, i)$ (with $a \neq b \in \{0, 1, \ldots, r-1\}^T$ and $1 \leq i \leq 2T$) consists of patterns $\begin{pmatrix} \beta_1 & \beta_2 \\ \alpha & \alpha \end{pmatrix}$ whose i-th column is the vector $\begin{pmatrix} b \\ a \end{pmatrix}$.

A $2 \times 2T$ matrix may fit multiple colour classes. When this occurs, we break the tie arbitrarily. The total number of colours are $r^T + r^2T$ (reduced from r^{3T}). By the definition of v and (10), one of the following cases must happen.

Case 1: There is a number $\alpha \in \{0, 1, \ldots, r-1\}^T$ such that there is a monochromatic clique of size $(r-1)(\ell-1)+1$ using colour $\begin{pmatrix} \alpha & \alpha \\ \alpha & \alpha \end{pmatrix}$.

We get the following $((r-1)(\ell-1)+1)$-rowed submatrix:

$$\begin{bmatrix} ** & \alpha\alpha & \alpha\alpha & \alpha\alpha & \cdots \\ \alpha\alpha & ** & \alpha\alpha & \alpha\alpha & \cdots \\ \alpha\alpha & \alpha\alpha & ** & \alpha\alpha & \cdots \\ \alpha\alpha & \alpha\alpha & \alpha\alpha & ** & \\ \vdots & \vdots & \vdots & & \end{bmatrix}$$

Since all the diagonal entry have two choices, we can pick one colour other than the corresponding colour in α. We get the following sub-matrix:

$$\begin{bmatrix} \beta_1 & \alpha & \alpha & \alpha & \cdots \\ \alpha & \beta_2 & \alpha & \alpha & \cdots \\ \alpha & \alpha & \beta_3 & \alpha & \cdots \\ \alpha & \alpha & \alpha & \beta_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

where the diagonal entry $\beta_i(j) \neq \alpha(j)$ for any i and any $j = 1, 2, \ldots, T$.

Using (8), the Pigeonhole principle yields a colour a appearing in α at least $(r-1)(t-1)+1$ times. By selecting these columns we get an $((r-1)(\ell-1)+1)$-rowed submatrix

$$\begin{bmatrix} *\cdots* & a\cdots a & a\cdots a & a\cdots a & \cdots \\ a\ldots a & *\cdots* & a\cdots a & a\cdots a & \cdots \\ a\cdots a & a\cdots a & *\cdots* & a\cdots a & \cdots \\ a\cdots a & a\cdots a & a\cdots a & *\cdots* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Note that all diagonal elements (marked by $*$) are not equal to a. By Pigeonhole principle, each diagonal entry has one colour $b_i \neq a$ appearing at least t times. Since the number of row s is $(r-1)(\ell-1)+1$, among those b_i's, there is a colour b appears in at least ℓ rows by Pigeonhole principle. This gives a configuration $t \cdot I_\ell(b, a)$, contradicting $A \in \text{Avoid}(m, r, t \cdot \mathcal{T}_\ell(r))$.

Case 2: There is a pair $a \neq b \in \{0, 1, \ldots, r-1\}$ and an index $i \in \{1, 2, \ldots, 2T\}$ such that there is a monochromatic clique of size $2t\ell$ using colour $C(a, b, i)$.

By selecting i-th column from each $1 \times 2T$ entry, we obtain a $2t\ell \times 2t\ell$-square submatrix of A so that the entry below the diagonal are all a's and the entry above the diagonal are all b's. The diagonal entries could be arbitrary. By deleting first column, second row, third column, fourth row, and so on, we get a submatrix $T_{t\ell}(a, b)$ of A and this contains $t \cdot T_\ell(a, b)$, contradicting $A \in \text{Avoid}(m, r, t \cdot \mathcal{T}_\ell(r))$.

Both cases end in a contradiction so we may conclude $|\{i\colon \|C_i\| \geq T\}| < v$. ■

Proof of Theorem 4.1: We consider $(t-1)$-simple matrices. Consider some $A \in \text{Avoid}(m, r, t \cdot \mathcal{T}_\ell(r))$ so that $A \in \text{Avoid}(m, r, t \cdot \mathcal{T}_\ell(r), t-1)$. Obtain the inductive structure of (7) with the bound (3).

By Lemma 4.4, $|\{i : \|C_i\| \geq T\}| \leq v$ with T given in (8). For each i, let C_i' denote the simple matrix obtained from C_i by reducing multiplicities to 1. Then $\|C_i'\| \leq \text{forb}(m-i, r, \mathcal{T}_\ell(r))$ since the multiplicity of each column α (of C_i) in $C_iC_iA_i$ is at least $s+1 \geq t$. By Theorem 1.2, there are at most $2^{c_r\ell^2}$ distinct columns in each C_i. Since C_i is $\frac{rs}{2}$-simple, we have

$$\|C_i\| \leq \frac{rs}{2} \cdot 2^{c_r\ell^2}.$$

We obtain using (3)

$$\begin{aligned}
\|A\| &\leq 2\left(\sum_{i\,:\,\|C_i\|<T} \|C_i\| + \sum_{i\,:\,\|C_i\|\geq T} \|C_i\|\right) + rs \\
&\leq 2(T-1)m + 2u \cdot \frac{rs}{2} \cdot 2^{c_r\ell^2} + rs \\
&= 2r(r-1)(t-1)m + c,
\end{aligned}$$

for a constant c depending on r, s, ℓ (note $s \geq t-1$). This is the desired bound, albeit with c being quite large. ■

5 Applications

We can apply our results in several ways. The following two variations for the problem of forbidden configurations are noted in [4].

Fixed Row order for Configurations: There have been some investigations for cases where only column permutations of are allowed. Note in our proof, the row order is fixed. So Theorem 1.2 works for this variation with the exact same upper bound.

Forbidden submatrices: When both row and column orders are fixed, this is the problem of forbidden submatrices. Let $I_\ell^R(a,b)$ and $T_\ell^R(a,b))$ be the matrix obtained from $I_\ell(a,b)$ and $T_\ell(a,b)$ respectively by reversing the column order. Our first observation is that any matrix in the four family $\{I_\ell(a,b)\}_{\ell\geq 3}$, $\{I_\ell^R(a,b)\}_{\ell\geq 3}$, $\{T_\ell(a,b)\}_{\ell\geq 3}$, and $\{T_\ell^R(a,b)\}_{\ell\geq 3}$ cannot be the submatrix of the one in another family. We have the following submatrix version of Theorem 1.2.

Theorem 5.1 *For any $r \geq 2$, there is a constant c_r so that for any $\ell \geq 2$ and any matrix with entries drawn from $\{0, 1, ..., r-1\}$ with at least $2^{c_r\ell^4}$ different columns must contain a submatrix $I_\ell(a,b)$, $T_\ell(a,b)$, $I_\ell^R(a,b)$, or $T_\ell^R(a,b)$ for some $a \neq b \in \{0, 1, ..., r-1\}$.*

Proof: Let $c_r = 30\binom{r}{2}^2 \log_2 r$. Let A be a simple r-matrix with $\|A\| > 2^{c_r \ell^4}$ and apply Theorem 1.2. Then A has an $\ell^2 \times \ell^2$ submatrix F which is a column permutation of $I_{\ell^2}(a,b)$ or $T_{\ell^2}(a,b)$. Let the column permutation be σ. By the fundamental Erdős-Szekeres Theorem, any sequence of $(\ell-1)^2+1 \le \ell^2$ distinct numbers must contain a monotone subsequence of ℓ numbers. Let $i_1 < i_2 < \cdots < i_\ell$ be the indexes so that the subsequence $\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_\ell)$ is either increasing or decreasing. Consider the submatrix of F obtained from F by restricting it to the $i_1, i_2, \ldots, i_\ell$ rows and $\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_\ell)$ columns. Then we obtained a submatrix which is one of the four matrices: $I_\ell(a,b)$, $T_\ell(a,b)$, $I_\ell^R(a,b)$, or $T_\ell^R(a,b)$. ■

We also can obtain some interesting variants of Theorem 1.2 by replacing some of the matrices in $\mathcal{T}_\ell(r)$. As noted in [7], we must forbid at least one (a,b)-matrix for each pair $a, b \in \{0, 1, \ldots, r-1\}$ in order to have a polynomial bound. What follows provides some additional examples of forbidden families related to $\mathcal{T}_\ell(r)$ with interesting polynomial bounds. We define $\text{forbmax}(m, r, \mathcal{F}) = \max_{m' \le m} \text{forb}(m', r, \mathcal{F})$. It has been conjectured that $\text{forbmax}(m, \mathcal{F}) = \text{forb}(m, \mathcal{F})$ for large m and for many $\mathcal{F}$ this can be proven. We have the following theorem.

Theorem 5.2 *Let r, ℓ be given and let $\pi = P_0 \cup P_1 \cup \cdots \cup P_{t-1}$ be a partition of $\{0, 1, \ldots, r-1\}$ into t parts. There is a constant $c_{\ell,r}$ such that for any family of matrices $\mathcal{F}_i$ all of whose entries lie in P_i $(1 \le i \le t)$*

$$forb(m, r, \left\{\mathcal{T}_{\ell,\pi}(r) \cup \bigcup_{i=0}^{t-1} \mathcal{F}_i\right\}) \le c_{\ell,r} \cdot \prod_{i=0}^{t-1} forbmax(m, |P_i|, \mathcal{F}_i).$$

$$\text{Here } \mathcal{T}_{\ell,\pi}(r) = \{I_\ell(a,b) \,:\, a \in P_i, b \in P_j, i \ne j\} \cup \{T_\ell(a,b) \,:\, a \in P_i, b \in P_j, i \ne j\}$$
$$\cup \{T_\ell(b,c) \,:\, b, c \in P_j, b \ne c\}.$$

Proof: Consider $A \in \text{Avoid}(m, r, \mathcal{T}_{\ell,\pi}(r) \cup \bigcup_{i=0}^{t-1} \mathcal{F}_i)$. Now form a t-matrix A_π from A by replacing each entry a of A that is in P_i by the entry i. Of course A_π is typically not simple but the maximum number of different columns is finite. Let $k = R_{r^2}(2\ell)$ and $c_{\ell,r} = 2^{c_t k^2}$ where c_t is the constant specified in Theorem 1.2. Otherwise A_π contains a configuration $I_k(i,j)$ or $T_k(i,j)$ for some $i \ne j \in \{0, 1, \ldots, t\}$. Now we return the colours i and j to the original colours in P_i and P_j respectively. We obtain a $k \times k$ matrix $F \prec A$ of one of the following two types.

type $I_k(i,j)$: All diagonal entries of F are in P_i while all off-diagonal entries are in P_j. First we apply the Pigeonhole principle on the diagonal and get a square submatrix F_1 of size $k/|P_i|$ so that all diagonal entries have the common value, say a, in P_i. Then we apply the multicolour Ramsey Theorem to F_1, where F_1 is viewed as a $|P_j|^2$-colouring of the complete graph on $k/|P_i|$ vertices with edge (x,y) (for $x < y$) coloured (b,c) if the (x,y) entry of F_1 is c and the (y,x) entry of F_1 is b . Since $k/|P_i| > R_{|P_j|^2}(2\ell)$, there exists a monochromatic clique of size 2ℓ in F_1. Say the colour is (b,c), where $b,c \in P_j$. If $b = c$, we obtain $I_{2\ell}(a,b)$ and so $I_\ell(a,b)$, a contradiction. If $b \neq c$, we obtained a $2\ell \times 2\ell$ matrix with a's on the diagonal, b's below the diagonal and c's above the diagonal. Forming the submatrix consisting of the odd indexed rows and the even indexed columns, we obtain $T_\ell(b,c)$, a contradiction.

type $T_k(i,j)$**:** All entries below diagonal of F are in P_i while the rest of entries are in P_j. We apply the multicolour Ramsey Theorem to F to obtain a submatrix F_2 of size 2ℓ whose lower-diagonal entries has a common value $a \in P_i$ and whose upper-diagonal entries has a common value $b \in P_j$. There is no restriction on the diagonal of F_2. We can get $T_\ell(a,b)$ from F_2 by deleting the first column, the second row, and so on. Again we have a contradiction.

Thus the number of different columns in A_π is bounded by a constant. Now consider $\mu(\alpha, A_\pi)$. If we replace just the i's in α by symbols chosen from P_i in more than forbmax$(m, |P_i|, \mathcal{F}_i)$ ways then we get some $F \in \mathcal{F}_i$ with $F \prec A$, a contradiction. So $\mu(\alpha, A_\pi) \leq \prod_{i=0}^{t-1}$ forbmax$(m, |P_i|, \mathcal{F}_i)$. Combined with our bound on the number of different columns in A_π, we are done. $\blacksquare$

Remark 5.3 *The constant $c_{\ell,r}$ in Theorem 5.2 is doubly exponential in ℓ in the proof above. One can reduce it into $2^{c'_r \ell^2}$ for some constant c'_r if we mimic the proof of Theorem 1.2 and use the Ramsey Theorem once. The details are omitted here.*

When applying it to the partition $\{0,1\} \cup \{2\} \cup \{3\} \cup \cdots \cup \{r\}$, we have the following theorem.

Theorem 5.4 *Let r, ℓ be given. There is a constant $c_{\ell,r}$ so that the following statement holds. Let $\mathcal{F}$ be a family of (0,1)-matrices. Then*

$$\mathit{forb}(m, r, (\mathcal{T}_\ell(r) \backslash \mathcal{T}_\ell(2)) \cup \{T_{\ell+1}(0,1)\} \cup \mathcal{F}) \leq c_{\ell,r} \cdot \mathit{forbmax}(m, s, \mathcal{F}).$$

This shows that, asymptotically at least, forbidding the configurations of $\mathcal{T}_\ell(r) \backslash \mathcal{T}_\ell(2) \cup T_{\ell+1}(0,1)$ is much like restricting us to (0,1)-matrices.

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