Forbidden Configurations: A Survey

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Introduction

Forbidden configurations are first described as a problem area in a 1985 paper. The subsequent work has involved a number of coauthors: Farzin Barekat, Laura Dunwoody, Ron Ferguson, Balin Fleming, Zoltan Füredi, Jerry Griggs, Nima Kamoosi, Peter Keevash and Attila Sali but there are works of other authors (some much older, some recent) impinging on this problem as well. For example, the definition of VC-dimension uses a forbidden configuration.

Survey at www.math.ubc.ca/∼anstee

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Definition We say $\mathcal{F} \subseteq 2^{[m]}$ is intersecting if for every pair $A, B \in \mathcal{F}$, we have $|A \cap B| \geq 1$.

Theorem *If $\mathcal{F} \subseteq 2^{[m]}$ and $\mathcal{F}$ is intersecting, then*

$$|\mathcal{F}| \leq 2^{m-1}.$$

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i.e. if A is m-rowed and simple then A is the incidence matrix of some $\mathcal{F} \subseteq 2^{[m]}$.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathcal{F} = \left\{\emptyset, \{2\}, \{3\}, \{1,3\}, \{1,2,3\}\right\}$$

Definition Given a matrix F, we say that A has F as a *configuration* if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} = A$$

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We consider the property of forbidding a configuration F in A for which we say F is a *forbidden configuration* in A.

Definition Let $\text{forb}(m, F)$ be the largest function of m and F so that there exist a $m \times \text{forb}(m, F)$ simple matrix with *no* configuration F. Thus if A is any $m \times (\text{forb}(m, F) + 1)$ simple matrix then A contains F as a configuration.

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For example, $\text{forb}(m, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = 2, \quad \text{forb}(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = m + 1.$

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows (i.e. incidence matrix of $2^{[k]}$).

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$\mathit{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots \binom{m}{0} = \Theta(m^{k-1})$$

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Theorem (Füredi 83). *Let F be a $k \times l$ matrix. Then $forb(m, F) = O(m^k)$*

Two interesting examples

Let $$F_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{forb}(m, F_1) = 2m, \qquad \text{forb}(m, F_2) = \left\lfloor \frac{m^2}{4} \right\rfloor + m + 1$$

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Problem *What drives the asymptotics of forb*(m,F)*? What structures in* F *are important?*

The building blocks of our constructions are I, I^c and T:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin I, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin I^c, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin T$$

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Note that $\mathrm{forb}(m, \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \mathrm{forb}(m, \begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \mathrm{forb}(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = m + 1$

Definition Given an $m_1 \times n_1$ matrix A and a $m_2 \times n_2$ matrix B we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of A on top of a column of B. If A, B are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \times \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} = \begin{bmatrix}1&1&1&0&0&0&0&0&0\\0&0&0&1&1&1&0&0&0\\0&0&0&0&0&0&1&1&1\\1&1&1&1&1&1&1&1&1\\0&1&1&0&1&1&0&1&1\\0&0&1&0&0&1&0&0&1\end{bmatrix}$$

Given p simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the p-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.

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The Conjecture

Definition Let $x(F)$ denote the largest p such that there is a p-fold product which does not contain F as a configuration where the p-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

Thus $x(F) + 1$ is the smallest value of p such that F is a configuration in every p-fold product $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

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The conjecture has been verified for $k \times l$ F where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $l = 2$ (A, Keevash 06) and other cases.

Theorem (Balogh, Bollabás 05) *Given k, there is a constant c_k with*

$$forb(m, \{I_k, I_k^c, T_k\}) = c_k.$$

Can we get good bounds on this constant?

Refinements of the Sauer Bound

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71) *forb*(m, K_k) *is* $\Theta(m^{k-1})$

Let $E_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Theorem (A, Fleming) *Let F be a $k \times l$ simple matrix such that there is a pair of rows with no configuration E_1 and there is a pair of rows with no configuration E_2 and there is a pair of rows with no configuration E_3. Then forb(m, F) is $O(m^{k-2})$.*

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Note that $F_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ has no E_1 on rows 1,3, no E_2 on rows 1,2 and no E_3 on rows 2,3. Thus forb(m, F_1) is $O(m)$.

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Theorem (A, Fleming) *Let E be given with $E \in \{E_1, E_2, E_3\}$. Let F be a $k \times l$ simple matrix with the property that every pair of rows contains the configuration E. Then forb$(m, F) = \Theta(m^{k-1})$.*

$$F_2 = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\0 & 0 & 1 & 1\end{bmatrix}$$ has E_3 on rows 1,2.

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Note that F_2 has E_3 on every pair of rows hence forb(m, F_2) is $\Theta(m^2)$ (A, Griggs, Sali 97).

In particular, this means $F_2 \notin T \times T$ which is the construction to achieve the bound.

Definition Let $t \cdot A$ be the matrix consisting of t copies of A placed side by side.

Theorem (Gronau 80)

$$forb(m, 2 \cdot K_k) = \binom{m}{k} + \binom{m}{k-1} + \cdots \binom{m}{0} = \Theta(m^k)$$

Theorem (A, Füredi 86)

$$forb(m, t \cdot K_k) = \frac{t-2}{k+1}\binom{m}{k}(1+o(1)) + \binom{m}{k} + \binom{m}{k-1} + \cdots \binom{m}{0}$$

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Theorem (A, Griggs, Sali 97, A, Sali 05, A, Fleming, Füredi, Sali 05)
forb$(m, F_B(t))$ *is* $\Theta(m^{k-1})$.

The difficult problem here was the bound although induction works.

Let A be given and let $S \in \binom{[m]}{k}$. We define $A|_S$ to be the submatrix of A given by the k rows indexed by S. For a given $k \times 1$ (0,1)-column α, use the notation

$\#\alpha$ in $A|_S$

to be the number of columns of $A|_S$ equal to α (not permuting rows).

Theorem (A, Fleming, Füredi, Sali 05) *Let A be an m-rowed simple matrix and let $\mathcal{S} \subseteq \binom{[m]}{k}$ where for each $S \in \mathcal{S}$ there are two $k \times 1$ (0,1)-columns α_S, β_S with*

$\#\alpha_S$ in $A|_S \le t$ and $\#\beta_S$ in $A|_S \le t$.

Then the number of columns γ in A such that there exists an $S \in \mathcal{S}$ with $\gamma|_S = \alpha_S$ or $\gamma|_S = \beta_S$, is at most $2t\left(\binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}\right)$.

Let $F_3(t) = \left[\begin{array}{c|c} \begin{matrix}0\\0\\0\end{matrix} & t \cdot \begin{bmatrix}0&0&0&1&1\\0&1&1&0&0\\1&0&1&0&1\end{bmatrix}\end{array}\right]$.

Theorem (A, Sali 05) *forb*$(m, F_3(t))$ *is* $\Theta(m^2)$.

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Let $F_k(t) = \left[\begin{array}{c|c} 0 & \\ \vdots & t \cdot G_k \\ 0 & \end{array}\right]$,

where G_k is the $k \times (2^k - 2^{k-2} - 1)$ simple matrix of all columns with at least one 1 and not having 1's in both rows 1 and 2 at the same time.

Conjecture (A, Sali 05) *$forb(m, F_k(t))$ is $\Theta(m^{k-1})$.*

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The standard induction does not work here.

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Theorem (A, Sali 05) *$forb(m, t \cdot I_k)$ is $\Theta(m^{k-1})$.*

Definition We say $\mathcal{F} \subseteq 2^{[m]}$ is t-intersecting if for every pair $A, B \in \mathcal{F}$, we have $|A \cap B| \geq t$.

Theorem (Katona 64) *Let t, m be given. Let $\mathcal{F} \subseteq 2^{[m]}$ be a maximum sized t-intersecting family. Then $\mathcal{F} = \mathcal{K}_{m,t}$ where*

$$\mathcal{K}_{m,t} = \begin{cases} \{A \subseteq [m] : |A| \geq (m+t)/2\} & \text{if } m+t \text{ is even} \\ \{A \subseteq [m] : |A \backslash \{1\}| \geq (m+t-1)/2\} & \text{if } m+t \text{ is odd} \end{cases}$$

Definition We say $\mathcal{F} \subseteq 2^{[m]}$ is *t-intersecting* if for every pair $A, B \in \mathcal{F}$, we have $|A \cap B| \geq t$.

Theorem (Ahlswede and Khachatrian 97)
Complete Intersection Theorem.
Let m, k, r be given. A maximum sized $(k\text{-}r)$-intersecting k-uniform family $\mathcal{F} \subseteq \binom{[m]}{k}$ is isomorphic to $\mathcal{I}_{r_1,r_2}$ for some choice $r_1 + r_2 = r$ and for some choice $G \subseteq [m]$ where $|G| = k - r_1 + r_2$ where
$\mathcal{I}_{r_1,r_2} = \{A \subseteq \binom{[m]}{k} : |A \cap G| \geq k - r_1\}$

This generalizes the Erdős-Ko-Rado Theorem (61).

Theorem (A-Keevash 06) Stability Lemma.
Let r *be given. Let* $\mathcal{F} \subseteq \binom{[m]}{k}$.
Assume that $\mathcal{F}$ *is* $(k\text{ - }r)$*-intersecting and*

$$|\mathcal{F}| \geq (6r)^{5r+7}m^{r-1}.$$

Then $\mathcal{F} \subseteq \mathcal{I}_{r_1,r_2}$ *for some choice* $r_1 + r_2 = r$ *and for some choice* $G \subseteq [m]$ *where* $|G| = k - r_1 + r_2$.

This result is for large intersections; we use it with a fixed r where k can grow with m.

Definition Let $F_{a,b,c,d}$ denote the $(a+b+c+d)\times 2$ matrix of a rows [11], b rows of [10], c rows of [01], and d rows of [00]. We assume $a \ge d$ and $b \ge c$.

Theorem (A-Keevash 06) *if* $b > c$ *or* $a, b \ge 1$, *then*

$$forb(m, F_{a,b,c,d}) = \Theta(m^{a+b-1}).$$

Also $forb(m, F_{0,b,b,0}) = \Theta(m^b)$ *and* $forb(m, F_{a,0,0,d}) = \Theta(m^a)$.

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Theorem (A-Keevash 06) *if* $b > c$ *or* $a, b \ge 1$, *then*

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Also $forb(m, F_{0,b,b,0}) = \Theta(m^b)$ *and* $forb(m, F_{a,0,0,d}) = \Theta(m^a)$.

Proof: The conjecture yields constructions. The proofs of the bounds make heavy use of the stability lemma in conjunction with induction.

Definition Let $F_{a,b,c,d}$ denote the $(a+b+c+d) \times 2$ matrix of a rows [11], b rows of [10], c rows of [01], and d rows of [00]. We assume $a \geq d$ and $b \geq c$.

Theorem (A-Keevash 06) *if $b > c$ or $a, b \geq 1$, then*

$$forb(m, F_{a,b,c,d}) = \Theta(m^{a+b-1}).$$

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Proof: The conjecture yields constructions. The proofs of the bounds make heavy use of the stability lemma in conjunction with induction.

The theorem is further evidence for the conjecture.

e.g. Let A be a simple matrix with no $F_{0,3,2,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} 0\,0 \cdots 0 & 1\,1 \cdots 1 \\ B_1 \quad B_2 & B_2 \quad B_3 \end{bmatrix}$ (the standard induction),
where B_2 is chosen to be all columns which are repeated after deleting row 1 of A.
Then $[B_1 B_2 B_3]$ is simple and has no $F_{0,3,2,0}$ and so by induction has at most $c(m-1)^2$ columns. B_2 is also simple and we verify that B_2 has at most cm columns and so by induction
$\text{forb}(m, F_{0,3,2,0}) \leq c(m-1)^2 + cm \leq cm^2$.

Given $A = \begin{bmatrix} 0\,0 \cdots 0 & 1\,1 \cdots 1 \\ B_1 \quad B_2 & B_2 \quad B_3 \end{bmatrix}$ with no $F_{0,3,2,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

then B_2 has no $F_{0,2,2,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ or $F_{0,3,1,0} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Our proof then uses the fact that if we only consider the columns of column sum k in B_2 as a set system, then using the fact that $F_{0,2,2,0}$ is forbidden we deduce that the k-uniform set system is $(k-1)$-intersecting. We then use our stability result to either determine the columns have a certain structure or that the bound is true because there are so few columns.

Some Exact Bounds

A, Griggs, Sali 97, A, Ferguson, Sali 01, A, Kamoosi 07
A, Barekat, Sali 08, A, Barekat 08

Configuration F	**Exact Bound forb(m, F)**
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	2
$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$	$m+2$
$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$2m+2$
$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$	$\left\lfloor \frac{5m}{2} \right\rfloor + 2$
$q \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\left\lfloor \frac{(q+1)m}{2} \right\rfloor + 2$, for m large

Configuration F	Exact Bound forb(m, F)
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\left\lfloor \frac{3m}{2} \right\rfloor + 1$
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$	$\left\lfloor \frac{7m}{3} \right\rfloor + 1$
$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\left\lfloor \frac{11m}{4} \right\rfloor + 1$
$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\left\lfloor \frac{15m}{4} \right\rfloor + 1$

Configuration F	Exact Bound forb(m, F)
$\begin{bmatrix}1&1&0&0&0\\0&0&1&1&1\end{bmatrix}$	$\left\lfloor \frac{8m}{3} \right\rfloor$
$\begin{bmatrix}1&1&0&0&0&0\\0&0&1&1&1&1\end{bmatrix}$	$\left\lfloor \frac{10m}{3} - \frac{4}{3} \right\rfloor$
$\begin{bmatrix}1&1&0&0&0&0&0\\0&0&1&1&1&1&1\end{bmatrix}$	$4m$
$\begin{bmatrix}\overbrace{1\cdots1}^{p}&\overbrace{0\cdots0}^{p}\\0\cdots0&1\cdots1\end{bmatrix}$	$pm - p + 2$

$$F_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Theorem (A, Dunwoody) $\text{forb}(m, F_2) = \lfloor \frac{m^2}{4} \rfloor + m + 1$

Proof: The proof technique is that of shifting, popularized by Frankl. A paper of Alon 83 using shifting refers to the possibility of such a result.

Configuration F	$\text{forb}(m, F)$
$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	$\binom{m}{2} + m - 2$
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\leq \frac{3}{2}\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$ $\geq \frac{29}{21}\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$ no exact coefficient of $\binom{m}{2}$ possible

Configuration F	$\mathbf{forb}(m, F)$
$\begin{array}{c} p \\ \overbrace{\hphantom{1\ 1\cdots 1}} \\ \left[\begin{array}{c} 1\ 1\cdots 1 \\ 1\ 1\cdots 1 \\ 0\ 0\cdots 0 \\ 0\ 0\cdots 0 \end{array}\right] \end{array}$	$\frac{p+3}{3}\binom{m}{2} + 2\binom{m}{1} + 2\binom{m}{0}$ for m large, $m \equiv 1, 3 (\text{mod } 6)$

THANKS FOR THE CHANCE TO VISIT COLLEGE OF CHARLESTON!