Forbidden Configurations: Progress on a Conjecture

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Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.

i.e. if A is m-rowed then A is the incidence matrix of some family $\mathcal{A}$ of subsets of $[m] = \{1, 2, \ldots, m\}$.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{A} = \{\emptyset, \{1,2,4\}, \{1,4\}, \{1,2\}, \{1,2,3\}, \{1,3\}\}$$

Definition We define $\|A\|$ to be the number of columns in A.

$$\|A\| = 6$$

Definition Given a matrix F, we say that A has F as a *configuration* if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & & 0 & \boxed{0} \end{bmatrix}$$

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We consider the property of forbidding a configuration F in A.

Definition Let

forb(m, F)= $\max\{\|A\| : A \ m\text{-rowed simple, no configuration } F\}$

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

$$forb(m, K_k) = \max\{\|A\| \,:\, A \text{ has } VC - \text{dimension } k-1\}$$

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Corollary *Let F be a $k \times \ell$ simple matrix. Then $forb(m, F) = O(m^{k-1})$.*

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Theorem (Füredi 83). *Let F be a $k \times \ell$ matrix. Then $forb(m, F) = O(m^k)$.*

A Product Construction

The building blocks of our product constructions are I, I^c and T, e.g:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Definition Given two matrices A, B, we define the product $A \times B$ as the matrix whose columns are obtained by placing a column of A on top of a column of B in all possible ways. (A, Griggs, Sali 97)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Given p simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the p-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m/p)^p$ i.e. $\Theta(m^p)$ columns.

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Examples

$$[01] \times [01] = K_2$$

$$\overbrace{[01] \times [01] \times \cdots \times [01]}^{k} = K_k$$

$I_{m/2} \times I_{m/2}$ is vertex-edge incidence matrix of $K_{m/2,m/2}$

The Conjecture

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Definition Let F be given. Let $x(F)$ denote the largest p such that there is a p-fold product which does not contain F as a configuration where the p-fold product is $A_1 \times A_2 \times \cdots \times A_p$ where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

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The conjecture has been verified for $k \times \ell$ F where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $\ell = 2$ (A, Keevash 06) and for k-rowed F with bounds $\Theta(m^{k-1})$ or $\Theta(m^k)$ (A, Fleming 10) plus other cases.

Forbidden Families are Difficult

Let G be a given graph. We define $\mathrm{ex}(m, G)$ to be the maximum number of edges in a graph on m vertices which has no subgraph isomorphic to G. Let F denote the vertex-edge incidence matrix of graph G. Then

$$forb(m, \left\{F, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}) = \mathrm{ex}(m, G) + m + 1.$$

Theorem (Balogh and Bollabás 05) *Given k, there exists a constant c_k so that $forb(m, \{I_k, I_k^c, T_k\}) = c_k$.*

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Theorem (A. and Meehan 11) *Let* p, k *be given with* $p \geq 3k$*.*
Let $F = [\mathbf{0}_k|I_k] \times [\mathbf{0}_k|T_k] \times [I_k^c|\mathbf{1}_k] \times K_{p-3k}$*.*
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Using a result of A. and Fleming 10, there are three simple column-maximal 4-rowed F for which $forb(m, F)$ is quadratic. Here is one example:

$$F_8 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

How can we repeat columns in F_8 and still have a quadratic bound? We note that repeating either the column of sum 1 or the column of sum 3 will result in a cubic lower bound. Thus we only consider taking multiple copies of the columns of sum 2.

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How can we repeat columns in F_8 and still have a quadratic bound? We note that repeating either the column of sum 1 or the column of sum 3 will result in a cubic lower bound. Thus we only consider taking multiple copies of the columns of sum 2. For a fixed t, let

$$F_8(t) = \left[\begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{matrix} \; t \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \right]$$

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Theorem (A, Raggi, Sali 09) *Let t be given. Then forb$(m, F_8(t))$ is $\Theta(m^2)$. Moreover $F_8(t)$ is a boundary case, namely for any column α not already present t times in $F_8(t)$, then forb$(m, [F_8(t)|\alpha])$ is $\Omega(m^3)$.*

The proof of the upper bound is currently a rather complicated induction with some directed graph arguments.

5×6 Simple Configuration with Quadratic bound

The Conjecture predicts nine 5-rowed simple matrices F which are boundary cases, namely $forb(m, F)$ is predicted to be $\Theta(m^2)$ and for any column α we have $forb(m, [F|\alpha])$ being $\Omega(m^3)$. Such F happen all to be 5×6 simple matrices and we have handled the following case.

$$F_7 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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All 6-rowed Configurations with Quadratic Bounds

$$G_{6\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Theorem (A,Raggi,Sali) *Let F be any 6-rowed configuration. Then forb(m, F) is $\Theta(m^2)$ if F is a configuration in $G_{6\times 3}$ and forb(m, F) is $\Omega(m^3)$ if F is not a configuration in $G_{6\times 3}$.*

Proof: We use induction and the bound for F_7.

Induction

Let A be an $m \times forb(m, F_7)$ simple matrix with no configuration F_7. We can select a row r and reorder rows and columns to obtain

$$A = \begin{matrix} \text{row } r \end{matrix} \left[\begin{array}{cccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & & C_r & C_r & & D_r \end{array} \right].$$

Induction

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Now $[B_r C_r D_r]$ is an $(m-1)$-rowed simple matrix with no configuration F_7. Also C_r is an $(m-1)$-rowed simple matrix with no configurations in $\mathcal{F}$ where $\mathcal{F}$ is derived from F_7.

C_r has no F in

$$\mathcal{F} = \left\{ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \right\}$$

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Then

$\|A\| = forb(m, F_7) = \|B_r C_r D_r\| + \|C_r\| \leq forb(m-1, F_7) + \|C_r\|$.

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Then

$\|A\| = forb(m, F_7) = \|B_r C_r D_r\| + \|C_r\| \leq forb(m-1, F_7) + \|C_r\|$.

To show $forb(m, F_7)$ is quadratic it would suffice to show $\|C_r\|$ is linear for some choice of r.

Repeated Induction

Let C_r be an $(m-1)$-rowed simple matrix with no configuration in $\mathcal{F}$. We can select a row s_i and reorder rows and columns to obtain

$$C_r = \begin{matrix}\text{row } s_i \\ \end{matrix}\begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ E_i & & G_i & G_i & & H_i \end{bmatrix}.$$

To show $\|C_r\|$ is linear it would suffice to show $\|G_i\|$ is bounded by a constant for some choice of s_i. Our proof shows that assuming $\|G_i\| \geq 8$ for all choices s_i results in a contradiction.

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We may choose s_1 and form L_1.
Then choose $s_2 \in L_1$ and form L_2.
Then choose $s_3 \in L_2$ and form L_3.
etc.
We can show the sets $L_1 \backslash s_2, L_2 \backslash s_3, L_3 \backslash s_4, \ldots$ are disjoint.
Assuming $\|G_i\| \geq 8$ for all choices s_i results in $|L_i \backslash s_{i+1}| \geq 3$ which yields a contradiction.

An unusual Bound

Theorem (A,Raggi,Sali 10) *forb*$(m, \{T_2 \times T_2, T_2 \times I_2, I_2 \times I_2\})$ *is* $\Theta(m^{3/2})$.

$$T_2 \times T_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \; T_2 \times I_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$I_2 \times I_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Induction

Let A be an $m \times forb(m, \mathcal{F})$ simple matrix with no configuration in $\mathcal{F} = \{T_2 \times T_2, T_2 \times I_2, I_2 \times I_2\}$. We can select a row r and reorder rows and columns to obtain

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To show $\|A\|$ is $O(m^{3/2})$ it would suffice to show $\|C_r\|$ is $O(m^{1/2})$ for some choice of r. Our proof shows that assuming $\|C_r\| > 36m^{1/2}$ for all choices r results in a contradiction.

THANKS to University of Victoria for hosting this conference!

We can extend K_4 and still have the same bound

$[K_4|\mathbf{1}_2\mathbf{0}_2] =$

$$\left[\begin{array}{cccccccccccccccc|c} 1&1&1&1&0&1&1&1&0&0&0&1&0&0&0&0&1\\ 1&1&1&0&1&1&0&0&1&1&0&0&1&0&0&0&1\\ 1&1&0&1&1&0&1&0&1&0&1&0&0&1&0&0&0\\ 1&0&1&1&1&0&0&1&0&1&1&0&0&0&1&0&0 \end{array}\right]$$

Theorem (A., Meehan) For $m \geq 5$, we have $forb(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = forb(m, K_4)$.

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$$\left[\begin{array}{cccccccccccccccc|c} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Theorem (A., Meehan) For $m \geq 5$, we have
$\mathit{forb}(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = \mathit{forb}(m, K_4)$.
We expect in fact that we could add many copies of the column $\mathbf{1}_2\mathbf{0}_2$ and obtain the same bound, albeit for larger values of m.