On Rigidity of Unit-Bar Frameworks

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Ethan White

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The following individuals certify that they have read, and recommend to the Faculty of Graduate and Postdoctoral Studies for acceptance, the thesis entitled:

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Examining Committee:

Richard Anstee

Supervisor

József Solymosi

Supervisor

Abstract

A framework in Euclidean space consists of a set of points called joints, and line segments connecting pairs of joints called bars. A framework is flexible if there exists a continuous motion of its joints such that all pairs of joints with a bar remain at a constant distance, but between at least one pair of joints not joined by a bar, the distance changes. For example, a square in the plane is not rigid since it can be deformed into a family of rhombi. This thesis is mainly concerned with infinitesimal motions. Loosely speaking, a framework is infinitesimally rigid if it does not wobble. One example is a motion of a single joint, where all other joints are unmoving, such that the movement of the one joint is perpendicular to all bars attached to it. The distances in an infinitesimal motion are preserved in the initial instant of motion. Infinitesimally rigid frameworks are rigid, and is an easier quality to verify, thereby making it a popular notion of rigidity to study among engineerings, architects, and mathematicians. We present infinitesimally rigid bipartite unit-bar frameworks in $\mathbb{R}^n$, and infinitesimally rigid bipartite frameworks in the plane with girth up to 12. Our constructions make use of the knight's graph; a graph such that vertices (joints) are squares of a chessboard and edges (bars) represent legal moves of the knight. We show that copies of the knight's graph can be assembled to create infinitesimally rigid frameworks in any dimension. Our constructions answer questions of Hiroshi Maehara.

Lay Summary

A framework is a mathematical model of a physical structure consisting of bars secured at joints. The Eiffel Tower is an example of bar and joint structure, since it is primarily composed of iron beams bolted together. A framework is rigid if it cannot be deformed into a different shape. For example, a square can be bent into a rhombus, and so is not rigid. Most rigid shapes we think of have triangles. Hiroshi Maehara posed the problem of determining rigid frameworks without triangles, or without any shape with any odd number of sides. We present rigid frameworks with this property.

Preface

Unless otherwise stated, the results after the first section of this thesis are original and a product of joint work with my supervisor Professor József Solymosi. Several of the results contained in this thesis have been submitted for publication.

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1 Background

1.1 Rigidity of physical structures

It is important to know if a bridge, tower, or toolshed will fall over in the presence of wind, loading, or gravity. Many factors contribute to the rigidity of structures including materials, foundation, and external forces. We will be concerned with the geometric component of rigidity, and will study it using a model known as a framework, which will be defined formally in the next section. Frameworks are mathematical approximations for structures made from bars and joints such as wooden trusses found in roofs, or bolted metal works such as the Forth Bridge in Scotland, or the Eiffel Tower in France.

The bars of a framework are fixed in length and cannot be bent. Information such as building material is not captured by a framework, and so plastic and steel structures are not differentiated. Moreover, in real life, any building material will deform when subjected to sufficient forces. Another difference between frameworks and real structures is that the joints of a framework are universally flexible, like the shoulder or hip of a person. Most structures are built with securely fastened and immovable joints. On the other hand, the bolts and welds of a real structure are often the weakest part, and so it is useful to know if the load of a structure will be borne by its beams or its bolts. In the latter case, a revised design might be in order.

Despite its obvious deficiencies, the framework model and properties of rigidity can be applicable in building design. In particular, infinitesimal motions of a framework, a concept that will be defined shortly, result in finite vibrations and deformations after construction. Frameworks also provide an accessible model to determine if a structure is over-braced. A structure will typically be overengineered with safety in mind, at the cost of heavier and costlier constructions. Applications of rigidity can be found in other scientific disciplines. In biology, the active sites of protein molecules (locations of chemical reactions) have been observed to be more rigid than elsewhere on the molecule, when it is modelled as a framework. In robotics, by applying rigidity theory, an operator of multiple robots can determine and reorganize their formation by only considering a small fraction of the relative distances between robots, as opposed to all position measurements.

1.2 Rigidity of frameworks

The following is one of the most important concepts of this thesis.

Definition 1.1. (Framework)

Let G be a simple graph on n vertices, and $V = \{1, 2, \ldots, n\}$, E be the sets of vertices and edges of G, respectively. A *framework* in $\mathbb{R}^d$ is a embedding of the vertices of G in $\mathbb{R}^d$. We denote the framework by $G(\mathbf{p})$, where $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n) \in \mathbb{R}^{dn}$ are the coordinates of the vertices. We call $\mathbf{p}$ a *placement*, and require that $\mathbf{p}_i \neq \mathbf{p}_j$ for all $i \neq j$. For each vertex $i \in V$ call $\mathbf{p}_i$ a *joint* of $G(\mathbf{p})$, and if $ij \in E$, call $\mathbf{p}_i\mathbf{p}_j$ a *bar*. □

The fundamental question of rigidity theory is to determine whether or not a framework is rigid. There are many different notions of rigidity, and several are defined in this section.

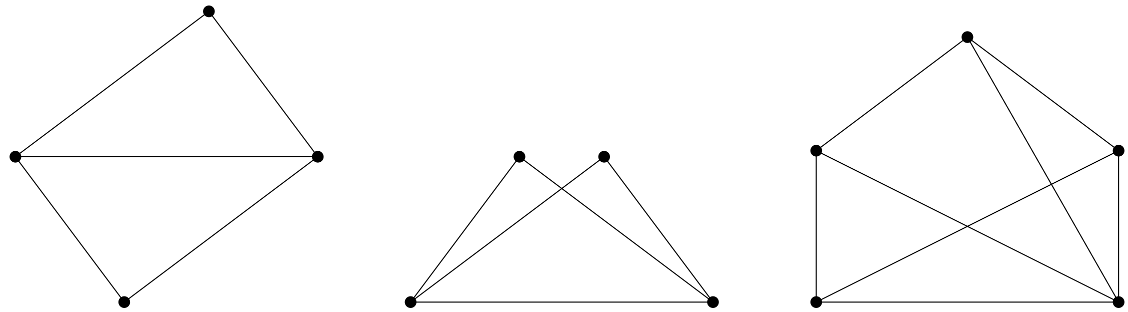
Definition 1.2. (Equivalent and Congruent) Two frameworks, $G(\mathbf{p})$ and $G(\mathbf{q})$ are called *equivalent* if for all edges $ij \in E$, we have

$$\|\mathbf{p}_i - \mathbf{p}_j\|^2 = \|\mathbf{q}_i - \mathbf{q}_j\|^2, \tag{1}$$

where $\|\cdot\|$ is the Euclidean norm. Denote equivalent frameworks by $G(\mathbf{p}) \equiv G(\mathbf{q})$. Furthermore, if (1) holds for all $i, j \in V$, then we say the two frameworks are *congruent*, and write $\mathbf{p} \equiv \mathbf{q}$. □

Definition 1.3. (Globally Rigid). A framework $G(\mathbf{p})$ is *globally rigid* if for all placements $\mathbf{q}$ of G such that $G(\mathbf{p}) \equiv G(\mathbf{q})$, then $\mathbf{p} \equiv \mathbf{q}$. □

The framework in Figure 1(a) is not globally rigid, since the framework in (b) is equivalent to it, but not congruent. It is easy to check that the framework in (c) is globally rigid. In [11], Saxe shows that determining if a framework is globally rigid in $\mathbb{R}$ is a strongly NP hard problem. We will see that other notions of rigidity result in more tractable problems.



(a) (b) (c)

Figure 1: Global rigidity in $\mathbb{R}^2$

Definition 1.4. (Continuously flexible). Let $G(\mathbf{p})$ be a framework in $\mathbb{R}^d$ with n joints. The framework $G(\mathbf{p})$ is *continuously flexible* if there exists a continuous path $\gamma\colon [0,1] \to \mathbb{R}^{dn}$ such that

i. $\gamma(0) = \mathbf{p}$

ii. $G(\gamma(t)) \equiv G(\mathbf{p})$ for all $t \in [0,1]$

iii. $\gamma(t) \not\equiv \mathbf{p}$ for $t > 0$.

We call γ a *flexion* of $G(\mathbf{p})$. A framework that is not continuously flexible, is *continuously rigid*. $\square$

This notion of rigidity most strongly agrees with the intuitive 'bent out of shape' definition. A triangle cannot be bent, since it's three side lengths uniquely determine it's shape. A square framework can be bent out of shape. For example, consider the framework with joints at $(0,0), (1,0), (0,1), (1,1)$ and four bars all length 1. Then

$$\gamma(t) = \begin{pmatrix} (0,0) \\ (1,0) \\ (\sin t, \cos t) \\ (\sin t + 1, \cos t) \end{pmatrix},$$

describes a flexion of the square, deforming it into a family of rhombi. Continuous rigidity can be formulated in several equivalent and useful ways.

Definition 1.5. (Neighbourhood rigid). A framework $G(\mathbf{p})$ is *neighbourhood rigid* if there exists $\epsilon > 0$ such that for all placements $\mathbf{q}$ of G with $\|\mathbf{p}_i - \mathbf{q}_i\| < \epsilon$ for all $i \in V$, and $G(\mathbf{p}) \equiv G(\mathbf{q})$, then $\mathbf{p} \equiv \mathbf{q}$. A framework that is not neighbourhood rigid is *neighbourhood flexible*. $\square$

A flexion of a framework $G(\mathbf{p})$ gives non-equivalent frameworks arbitrarily close to $G(\mathbf{p})$, and so continuous flexibility implies neighbourhood flexibility. On the other hand, let $X \subset \mathbb{R}^{dn}$ be the set of all placements such that $G(\mathbf{p}) \equiv G(\mathbf{q})$, and let $Y \subset \mathbb{R}^{dn}$ be the set of all placements such that $\mathbf{p} \equiv \mathbf{q}$. Note that X is an algebraic variety, and Y is a subvariety. Suppose that $G(\mathbf{p})$ is neighbourhood flexible. Then for every neighbourhood U of $\mathbf{p}$, there exists $\mathbf{q} \in U \cap (X - Y)$.

Remark 1.6. As a result of Lemma 1.7 below, there is an analytic path γ between $\mathbf{p}$ and $\mathbf{q}$, such that γ is a flexion. It follows that continuous and neighbourhood rigidity are equivalent. $\square$

Lemma 1.7. (Wallace [13]). Let V be a real algebraic variety, W a subvariety, and $\mathbf{p}$ a point of W. Then there is a neighbourhood U of $\mathbf{p}$ such that all points of $U \cap (V - W)$ can be jointed to $\mathbf{p}$ by analytic arcs on V meeting W only at $\mathbf{p}$. $\square$

This means that the requirement that flexions must be continuous is equivalent to condition they are infinitely differentiable. Let $\gamma = (\gamma_1, \ldots, \gamma_n)\colon [0,1] \to \mathbb{R}^{dn}$ be an analytic flexion of a framework $G(\mathbf{p})$. The size of a bar does not change under γ, hence for each edge $ij \in E$ we have

$$\frac{d}{dt}\|\gamma_i(t) - \gamma_j(t)\|^2 = 2(\gamma_i(t) - \gamma_j(t)) \cdot (\gamma_i'(t) - \gamma_j'(t)) = 0. \tag{2}$$

This means that the relative motion of two joints is perpendicular to the direction of a bar between them. This motivates another definition of rigidity.

Definition 1.8. (Infinitesimal rigidity). An *infinitesimal motion* of $\mathbb{R}^d$ is a vector field $f\colon \mathbb{R}^d \to \mathbb{R}^d$ such that for all points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$:

$$(\mathbf{x} - \mathbf{y}) \cdot (f(\mathbf{x}) - f(\mathbf{y})) = 0. \tag{3}$$

An *infinitesimal motion of a framework* $G(\mathbf{p})$ is a vector field $f\colon V \to \mathbb{R}^d$ that satisfies (3) for all bars $\mathbf{xy}$ of $G(\mathbf{p})$. An infinitesimal motion of a framework is *trivial* if it is the restriction of an infinitesimal motion of $\mathbb{R}^d$. A framework that admits a non-trivial infinitesimal motion is called *infinitesimally flexible*, otherwise the framework is *infinitesimally rigid*. $\square$

Intuitively, infinitesimal flexibility describes a frameworks ability to 'wobble'. From (2), we see an analytic flexion γ gives rise to an infinitesimal motion defined by $f(\mathbf{p}_i) = \gamma_i'(0)$. It's easy to check that f is non-trivial since (2) is nonzero for some $ij \notin E$. It follows that continuously flexible frameworks are infinitesimally flexible, and so infinitesimal rigidity implies continuous rigidity. On the other hand, continuous rigidity does not imply infinitesimal rigidity, see Figure 2 for an example.

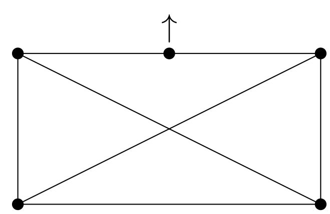


Figure 2: Continuously but not infinitesimally rigid framework in $\mathbb{R}^2$

The infinitesimal motion of the above framework that gives zero velocity to the four corner joints, but upwards velocity to the middle joint is non-trivial. The framework in Figure 2 is globally rigid also. On the other hand, the framework in Figure 1(a) is infinitesimally rigid, but not globally rigid. In summary, global and infinitesimal rigidity imply continuous rigidity, but neither implies the other.

Infinitesimal rigidity also captures the real-world intuition that 'flat' structures are not strong. For example, the framework of three joints pairwise connected by bars in $\mathbb{R}$ is infinitesimally rigid. But in $\mathbb{R}^2$, three collinear joints, pairwise connected by bars is infinitesimally flexible. In general, suitably 'flat' frameworks are infinitesimally flexible.

Proposition 1.9. Let F be a framework with n joints contained in the span of the basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_k$ in $\mathbb{R}^d$, where $k \leq n-2$, and $k \leq d-1$. Then F is infinitesimally flexible.

Proof. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$, be the joints of F. Without loss of generality, suppose $\mathbf{x}_1 = \mathbf{0}$. Define a function $f \colon \mathbb{R}^d \to \mathbb{R}^d$ such that $f(\mathbf{x}_i) = \mathbf{0}$ for $2 \le i \le n$, and $f(\mathbf{0}) = \mathbf{e}_{k+1}$. It is easy to check that f is an infinitesimal motion of F. On the other hand, suppose for a contradiction that f can be extended to an infinitesimal motion of $\mathbb{R}^d$. Then $(\mathbf{e}_{k+1} - \mathbf{x}_i) \cdot f(\mathbf{e}_{k+1})) = 0$ for all $2 \le i \le n$. Since $\{\mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is contained in a subspace of dimension at most $n-2$, they are linearly dependent. We can find $a_i \in \mathbb{R}$, $2 \le i \le n$ such that $\sum_{i=2}^n a_i \mathbf{x}_i = 0$ and $\sum_{i=2}^n a_i = 1$. Then $\sum_{i=2}^n a_i(\mathbf{e}_{k+1} - \mathbf{x}_i) = \mathbf{e}_{k+1}$. This implies $\mathbf{e}_{k+1} \cdot f(\mathbf{e}_{k+1}) = 0$ and so

$$(\mathbf{0} - \mathbf{e}_{k+1}) \cdot (\mathbf{e}_{k+1} - f(\mathbf{e}_{k+1})) = -\mathbf{e}_{k+1} \cdot \mathbf{e}_{k+1} = -1.$$

This is a contradiction, since the above must be 0 for an infinitesimal motion f of $\mathbb{R}^d$. It follows that F is infinitesimally flexible.

□

The framework in Figure 3 is infinitesimally rigid in $\mathbb{R}^2$, i.e. if it's joints are at coordinates $(0,0), (0,1), (1,0), (1,1)$. On the other hand if the same framework is in $\mathbb{R}^3$, with joints at $(0,0,0), (0,1,0), (1,0,0), (1,1,0)$ then it is infinitesimally flexible by Proposition 1.9.

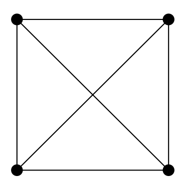


Figure 3: Infinitesimally rigid in $\mathbb{R}^2$, infinitesimally flexible in $\mathbb{R}^3$

It is easy to check that an infinitesimal motion f of $\mathbb{R}^d$ is determined by its values at the basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_d$. Since the set of infinitesimal motions is closed under addition and scalar multiplication, it can be viewed as a vector subspace of $\mathbb{R}^{d^2}$. Applying (3) to each pair $\mathbf{e}_i, \mathbf{e}_j$ gives $\binom{d}{2}$ independent constraints, and so the space of infinitesimal motions has dimension at most $d^2 - \binom{d}{2} = \binom{d+1}{2}$. Equation (2) shows that all smooth motions of $\mathbb{R}^d$ that are distance preserving, i.e. rotations and translations, give rise to infinitesimal motions. The space of orientation preserving rotations are the special orthogonal matrices and have dimension $\frac{(d-1)d}{2}$. The space of translations has dimension d. It follows that the space of infinitesimal motions of $\mathbb{R}^d$ has dimension $d + \frac{(d-1)d}{2} = \binom{d+1}{2}$. The infinitesimal motions of a framework can be described by the kernel of a matrix.

Definition 1.10. (Rigidity matrix). Let $G(\mathbf{p})$ be a framework in $\mathbb{R}^d$ with n joints and m bars. The *rigidity matrix* of $G(\mathbf{p})$ has a row for each bar, and d columns for each joint, corresponding to each coordinate of the joint. The matrix takes the form

$$R_{G(\mathbf{p})} = \begin{array}{c} \\ \text{edge } ij \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} & \text{vertex } i & & \text{vertex } j & \end{array} \\ \begin{pmatrix} & & & & \\ \cdots & \mathbf{p}_i - \mathbf{p}_j & \cdots & \mathbf{p}_j - \mathbf{p}_i & \cdots \\ & & & & \end{pmatrix} \end{array}$$

□

Each row of the rigidity matrix corresponds with the condition described in (3). It is easy to check that an infinitesimal motion of $\mathbb{R}^d$ that is zero on a set of points that are not contained in a $(d-2)$-dimensional hyperplane, is identically zero. As a result, if $G(\mathbf{p})$ is in $\mathbb{R}^d$ with joints not contained in a $(d-2)$-dimensional hyperplane, then distinct infinitesimal motions of $\mathbb{R}^d$ restrict to distinct infinitesimal motions of $G(\mathbf{p})$ and so the dimension of infinitesimal motions of $G(\mathbf{p})$ is at least $\binom{d+1}{2}$. Equivalently, this is

$$\text{rank } R_{G(\mathbf{p})} \leq dn - \binom{d+1}{2}, \tag{4}$$

and equality holds just in case $G(\mathbf{p})$ is infinitesimally rigid. The above is summarized in the following key theorem.

Theorem 1.11. (Rigidity matrix). Let $G(\mathbf{p})$ be a framework in $\mathbb{R}^d$ with at least d joints. Then $G(\mathbf{p})$ is infinitesimally rigid if and only if

$$\text{rank } R_{G(\mathbf{p})} = dn - \binom{d+1}{2}.$$ □

Proof. If $G(\mathbf{p})$ is not contained in a $(d-2)$-dimensional hyperplane, then by the above the result follows. Suppose on the other hand, that $G(\mathbf{p})$ is contained in a k-dimensional hyperplane, where $k \leq d-2$, and that $G(\mathbf{p})$. By translations and rotations, we can assume that $G(\mathbf{p})$ is contained in the subspace spanned by $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$. Let $G(\mathbf{p}')$ be the framework in $\mathbb{R}^k$ corresponding to the natural projection of taking the first k coordinates (note that the underlying graph will be the same). The rigidity matrices $R_{G(\mathbf{p})}$ and $R_{G(\mathbf{p}')}$ differ only by $(d-k)n$ columns of zeros, and so they have the same rank. By the above discussion,

$$\text{rank } R_{G(\mathbf{p}')} \leq kn - \binom{k+1}{2} < dn - \binom{d+1}{2}.$$

This proves the result. □

Note that the condition $n \geq d$ is necessary for (4) to hold. For example, the triangle framework with joints $(0,0,0,0)$, $(1,0,0,0)$, $(0,1,0,0)$ is infinitesimally rigid, and the rank of the corresponding rigidity matrix is 3. On the other hand, the quantity on the righthand side of (4) in this case is 2.

Example 1.12. Reconsider the framework in Figure 2, with the placement described in Figure 4.

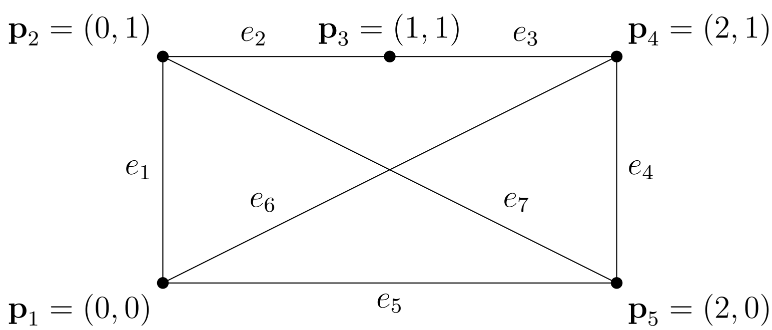


Figure 4: Infinitesimally flexible framework in $\mathbb{R}^2$

We calculate the rigidity matrix below using Definition 1.10.

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} \mathbf{p}_1 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_3 & \mathbf{p}_4 & \mathbf{p}_4 & \mathbf{p}_5 & \mathbf{p}_5 \end{array} \\
\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{array} &
\begin{pmatrix}
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
-2 & -1 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 2 & -1
\end{pmatrix}
\end{array}
\tag{5}
$$

The framework in question has at least two joints, and so the result of Theorem 1.11 applies. It is easy to check that the rank of the matrix above is 6, and so the framework is infinitesimally flexible. For a different perspective, recall that any infinitesimal motion of the plane in which two distinct joints are fixed is trivial, i.e. in this case all joints are fixed. The vector $\mathbf{v} = (0,0,0,0,0,1,0,0,0,0)$ is in the nullspace of the above matrix and hence corresponds to an infinitesimal motion of the framework. The motion described by $\mathbf{v}$ fixes the joints $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_4$, and $\mathbf{p}_5$, but moves $\mathbf{p}_3$, and so cannot be extended to an infinitesimal motion of the plane. $\square$

Example 1.13. Consider the framework of Figure 5.

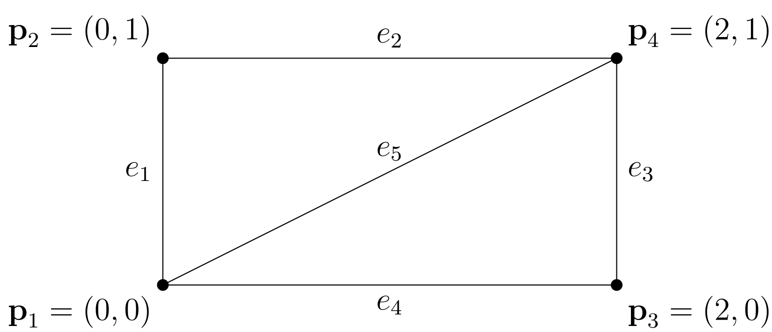


Figure 5: Infinitesimally rigid in $\mathbb{R}^2$

The rigidity matrix is

$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} \mathbf{p}_1 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_3 & \mathbf{p}_4 & \mathbf{p}_4 \end{array} \\
\begin{pmatrix}
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 \\
-2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
-2 & -1 & 0 & 0 & 0 & 0 & 2 & 1
\end{pmatrix}
\end{array}$$

One can check that the rank of the above matrix is 5, and so the framework is infinitesimally rigid in $\mathbb{R}^2$ (as expected). If any bar of the framework is removed, the rank of the matrix would decrease since it only has 5 rows, and the resulting framework would be infinitesimally flexible. $\square$

In general, an infinitesimally rigid framework in $\mathbb{R}^d$ must have at least $dn - \binom{d+1}{2}$ bars. We make this statement more precise below.

Proposition 1.14. If $G(\mathbf{p})$ is an infinitesimally rigid framework in $\mathbb{R}^d$ with n joints and m bars, then

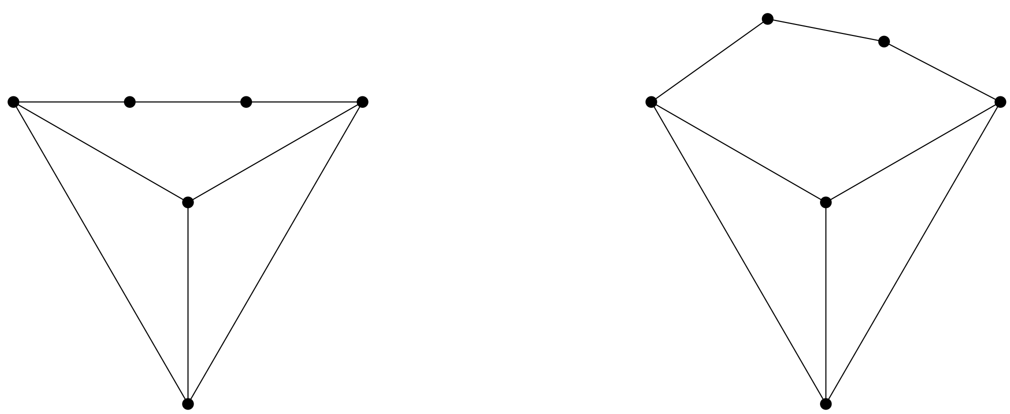
$$m \geq \begin{cases} dn - \binom{d+1}{2} & \text{if } n \geq d \\ \binom{n}{2} & \text{if } n \leq d-1 \end{cases}$$

Proof. The case $n \geq d$ is clear by Theorem 1.11. Suppose for a contradiction that $n \leq d-1$ and $m < \binom{n}{2}$, and so there is a pair of joints without a bar between them. Let $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ be the joints of $G(\mathbf{p})$ and suppose there is no bar between $\mathbf{p}_1$ and $\mathbf{p}_2$. Define a function f on the joints to $\mathbb{R}^d$ by $f(\mathbf{p}_i) = \mathbf{0}$ for $2 \leq i \leq n$. Consider that $U = \{\mathbf{u} \in \mathbb{R}^d \colon (\mathbf{p}_1 - \mathbf{p}_i) \cdot \mathbf{u} = 0,\ 3 \leq i \leq n\}$, is a vector subspace of $\mathbb{R}^d$ with dimension at least 2. Hence there exists $\mathbf{u} \in U$ such that if $f(\mathbf{p}_1) = \mathbf{u}$ then f is an infinitesimal motion of $G(\mathbf{p})$ but also $(\mathbf{p}_1 - \mathbf{p}_2) \cdot (f(\mathbf{p}_1) - f(\mathbf{p}_2)) \neq 0$. It follows that f cannot be extended to an infinitesimal motion of $\mathbb{R}^d$, and $G(\mathbf{p})$ is infinitesimally flexible, a contradiction.

$\square$

Notice that the two lower bounds in Proposition 1.14 are equal if $n = d$. Furthermore, if $n \leq d$, then $\binom{n}{2} \geq dn - \binom{d+1}{2}$, and so the number of bars in an infinitesimally rigid framework is always at least $dn - \binom{d+1}{2}$.

A continuously rigid framework need not posses at least $dn - \binom{d+1}{2}$ bars. See Figure 6(a) for an example. The framework in Figure 6(a) is rigid since the upper edges have no slack. The same graph is shown in Figure 6(b) with a more typical embedding, and is flexible. On the other hand, Figure 7 shows the two frameworks from the same graph, the framework on the left is continuously flexible, and the framework on right is infinitesimally rigid. The framework in Figure 7(b) is more typical of most placements of this graph. The rigidity of the framework in Figure 6(a), and flexibility of the framework in Figure 7(a), is 'nontypical'. This motivates another type of rigidity.



(a) (b)

Figure 6: Generically flexible in the plane

Definition 1.15. (Generic rigidity). A set of points in $\mathbb{R}^d$ is *generic* if the coordinates of the points are algebraically independent over the rationals. A framework $G(\mathbf{p})$ is generic if its joints $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \subset \mathbb{R}^d$ form a generic set. A graph G is *generically rigid* in $\mathbb{R}^d$ if there exists a placement $\mathbf{p}$ of its vertices such that $G(\mathbf{p})$ is infinitesimally rigid.

If a graph G is generically rigid, then all generic frameworks $G(\mathbf{p})$ are infinitesimally rigid. Moreover, any infinitesimal motion of a generic framework comes from the initial velocities

(a) (b)

Figure 7: Generically rigid in the plane

of a flexion, i.e. infinitesimal and continuous rigidity are equivalent for generic frameworks. Clearly, generic rigidity is a combinatorial property of the underlying graph. The following theorem describes the conditions for when a graph in the plane is generically rigid.

Theorem 1.16. (Laman's Theorem). A graph G is generically rigid in the plane if and only if for every subgraph $H \subset G$, with at least 2 vertices,

$$|E(H)| \leq d|V(H)| - \binom{d+1}{2}, \tag{6}$$

where above $d = 2$ and equality holds for $H = G$.

For any dimension, (6) is known as Maxwell's condition, and is a necessary condition for the rows of the rigidity matrix to be independent. In dimensions $d \geq 3$, Maxwell's condition is not sufficient. A well-known example of a flexible framework satisfying Maxwell's conditions in $d = 3$ is the 'double-banana' framework. This can be constructed by glueing two K_5's along an edge, and then deleting that edge. In general, no characterization of generically rigid graphs for dimensions more than two is known.

1.3 Unit-bar frameworks

The results of this thesis are concerned with frameworks in which all bars have unit length. Since properties of (infinitesimal) rigidity are preserved under scaling, we make the following definition.

Definition 1.17. A framework is a *unit-bar framework* if all bars have the same length. □

Any framework with two bars of the same length is not generic, and so the rigidity of unit-bar frameworks is a geometric property as well as a combinatorial one. In particular, Laman's theorem is not applicable to unit-bar frameworks. It is easy to construct rigid unit-bar frameworks in the plane by glueing together equilateral triangles, and in general by glueing d-simplexes in $\mathbb{R}^d$. Constructing rigid frameworks without triangles, or no odd-cycles is a more difficult problem.

Maehara et al. [7, 8, 9, 10] explore triangle-free and bipartite rigid unit-bar frameworks. In [7], Maehara found a rigid bipartite framework in $\mathbb{R}^2$ consisting of 353 joints and 676 unit-bars. It is not infinitesimally rigid. An infinitesimally rigid unit-bar triangle-free framework in $\mathbb{R}^2$ was given by Maehara and Chinen in [8]. Their framework has 22 joints and 41 unit-bars and is pictured in Figure 8.

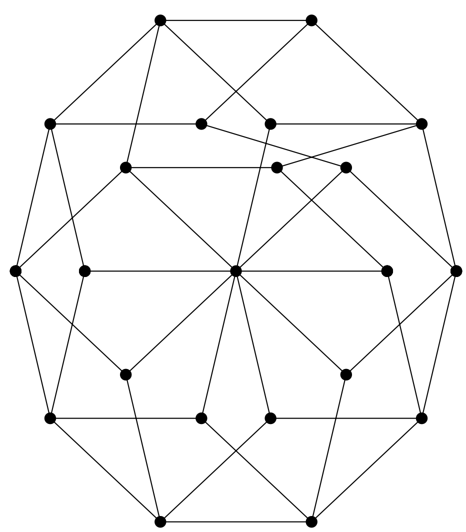


Figure 8: Maehara and Chinen's framework

An infinitesimally rigid unit-bar triangle-free framework in $\mathbb{R}^3$ with 26 joints and 78 unit-bars was found by Maehara and Tokushige, see [9]. Their framework is built beginning with a unit cube, and then attaching pairs of joints that are unit distance from some co-circular four tuples of joints of the cube. The frameworks in [8, 9] contain pentagons. In [8] and [10] the authors propose the following problems:

I. Find an infinitesimally rigid bipartite unit-bar framework in the plane

II. Find a general method to construct a triangle-free, infinitesimally rigid unit-bar framework in $\mathbb{R}^d$.

The main results of this thesis is the resolution of these problems.

1.4 Connections to the unit distance problem

The *unit distance problem* was posed by Paul Erdős in 1946: "how many pairs of n points in the plane can be unit distance apart?" [4]. This is one of the central open problems in discrete geometry. Erdős gave a construction that proved there are at least $n^{1+c/\log\log n}$ such pairs, and he conjectured that this is the true order of magnitude. The construction is as follows. There are infinitely many n such that the number of solutions to $n = p^2 + q^2$ is greater than $n^{c/\log\log n}$ [3]. For any such n, consider the points on a $2\lfloor\sqrt{n}\rfloor \times 2\lfloor\sqrt{n}\rfloor$ lattice. It is easy to check that there are at least $n^{1+c/\log\log n}$ pairs of points distance $\sqrt{n}$ apart. By scaling the lattice, these pairs of points are unit distance apart.

My supervisor, Jozsef Solymosi, noticed that Erdős' lower bound construction for the unit distance problem is a good candidate for rigid unit-bar frameworks. We will present several infinitesimally rigid frameworks that arise from Erdős' construction. In particular, using numbers that can be written as a sum of two squares in several ways, and a random computer algorithm, we construct infinitesimally rigid unit-bar frameworks on lattices with girth up to 12.

2 Bipartite Unit-Bar Frameworks in the Plane.

The results of this section correspond with Section 2 of [12]. Let $m, n, d \in \mathbb{N}$, and define $F_{m,n}(d)$ to be the framework in $\mathbb{R}^2$ with joint set $\{(x, y) \in \mathbb{Z}^2 \colon 0 \le x \le m-1, 0 \le y \le n-1\}$, and bars between all pairs of joints distance $\sqrt{d}$ apart. For odd d, two numbers summing to

d have different parity. Hence the sum of the coordinates of adjacent joints is different, and the framework is bipartite. One can show that for even d a framework constructed in this way is also bipartite, see for example [2].

The number of bars in $F_{m,n}(1)$ is $(m-1)n + m(n-1)$, the number of bars in $F_{m,n}(2)$ is $2(m-1)(n-1)$, and if $m, n \geq 4$, the number of bars in $F_{m,n}(4)$ is $(m-4)n + m(n-4)$. In all of these cases, there is no size of lattice with enough bars to be infinitesimally rigid, i.e. there is always less that $2mn - 3$ bars. On the other hand, if $m, n \geq 2$, the number of bars in $F_{m,n}(5)$ is $2(m-1)(n-2) + 2(m-2)(n-1)$. The smallest framework $F_{m,n}(5)$, with enough bars is $F_{5,5}(5)$. We will prove that $F_{5,5}(5)$ is infinitesimally rigid. The graph underlying $F_{5,5}(5)$ is the knight's graph, and so we refer to it as the 5×5 knight's framework, see Figure 9 below.

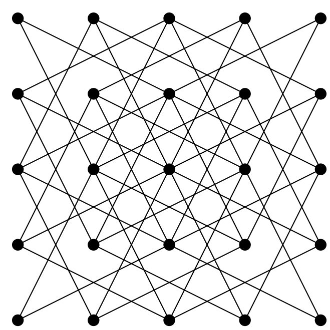
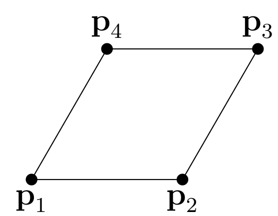


Figure 9: 5×5 knight's framework

2.1 Knight's Framework

The infinitesimal rigidity of the 5×5 knight's framework can be verified by calculating the rank of its rigidity matrix. The program in Appendix 2 does this calculation. The rank can also be computed without computer aid; however, it is a system of 50 variables. The following lemma reduces the number of variables and facilitates a shorter by-hand proof of the infinitesimal rigidity of $F_{5,5}(5)$.

Lemma 2.1. (Rhombus Lemma) Let $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4$ be a framework of a non-degenerate rhombus in the plane. If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are the velocity vectors associated with any infinitesimal motion of the rhombus, then $\mathbf{v}_1 + \mathbf{v}_3 = \mathbf{v}_2 + \mathbf{v}_4$.



Proof. Put $\mathbf{x} = \mathbf{p}_2 - \mathbf{p}_1 = \mathbf{p}_3 - \mathbf{p}_4$ and $\mathbf{y} = \mathbf{p}_3 - \mathbf{p}_2 = \mathbf{p}_4 - \mathbf{p}_1$. We have:

$$\begin{aligned}(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{x} &= 0,\\ (\mathbf{v}_3 - \mathbf{v}_2) \cdot \mathbf{y} &= 0,\\ (\mathbf{v}_4 - \mathbf{v}_3) \cdot \mathbf{x} &= 0,\\ (\mathbf{v}_1 - \mathbf{v}_4) \cdot \mathbf{y} &= 0.\end{aligned}$$

The first and third equation give $(\mathbf{v}_1 + \mathbf{v}_3) \cdot \mathbf{x} = (\mathbf{v}_2 + \mathbf{v}_4) \cdot \mathbf{x}$, while the second and fourth give $(\mathbf{v}_1 + \mathbf{v}_3) \cdot \mathbf{y} = (\mathbf{v}_2 + \mathbf{v}_4) \cdot \mathbf{y}$. Since $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, we have the desired result. □

If $f : \mathbb{R}^d \to \mathbb{R}^d$ is a function, let $f_k(\mathbf{x})$ denote the value in the k^{th} coordinate of $f(\mathbf{x})$.

Theorem 2.2. *The* 5×5 *knight's framework is infinitesimally rigid in* $\mathbb{R}^2$.

Proof. Let the joints of N_5 from left to right, top to bottom be $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{25}$. Consider all infinitesimal motions f of N_5 such that

$$f(\mathbf{p}_{13}) = f_1(\mathbf{p}_2) = 0. \tag{7}$$

This specifies three degrees of freedom of f, so the dimension of the space of infinitesimal motions of N_5 that satisfy (7) is at most three less than the dimension of the space of all infinitesimal motions of N_5. Since the space of infinitesimal motions of the plane has dimension three, if all infinitesimal motions f of N_5 that satisfy (7) are identically zero then N_5 is infinitesimally rigid. Let f be an infinitesimal motion of N_5 satisfying (7) and put $f(\mathbf{p}_i) = \mathbf{v}_i$ for all i. Since $\mathbf{p}_2\mathbf{p}_{13}$ is a bar we have that $\mathbf{v}_2 = 0$. Using Lemma 2.1, we are able to determine all velocities v_i homogenously in terms of the velocities $\mathbf{v}_4, \mathbf{v}_6, \mathbf{v}_{10}, \mathbf{v}_{20}$, and $\mathbf{v}_{22}$. The first equation in every line below follows from an application of Lemma 2.1 to a rhombus in N_5, a second equation in any line is a substitution of a previous equation. We have:

$$\mathbf{v}_3 = \mathbf{v}_6 + \mathbf{v}_{10}$$

$$\mathbf{v}_{11} = \mathbf{v}_{22}$$

$$\mathbf{v}_{15} = \mathbf{v}_4 + \mathbf{v}_{24}$$

$$\mathbf{v}_{23} = \mathbf{v}_{16} + \mathbf{v}_{20}$$

$$\mathbf{v}_7 = \mathbf{v}_4 + \mathbf{v}_{16}$$

$$\mathbf{v}_9 = \mathbf{v}_{20}$$

$$\mathbf{v}_{17} = \mathbf{v}_6 + \mathbf{v}_{24}$$

$$\mathbf{v}_{19} = \mathbf{v}_{10} + \mathbf{v}_{22}$$

$$\mathbf{v}_8 = \mathbf{v}_{11} + \mathbf{v}_{19} - \mathbf{v}_{22} = \mathbf{v}_{10} + \mathbf{v}_{22} \tag{8}$$

$$\mathbf{v}_8 = \mathbf{v}_{15} + \mathbf{v}_{17} - \mathbf{v}_{24} = \mathbf{v}_4 + \mathbf{v}_6 + \mathbf{v}_{24} \tag{9}$$

$$\mathbf{v}_{12} = \mathbf{v}_9 + \mathbf{v}_{23} - \mathbf{v}_{20} = \mathbf{v}_{16} + \mathbf{v}_{20} \tag{10}$$

$$\mathbf{v}_{12} = \mathbf{v}_3 + \mathbf{v}_{19} - \mathbf{v}_{10} = \mathbf{v}_6 + \mathbf{v}_{10} + \mathbf{v}_{22} \tag{11}$$

$$\mathbf{v}_{14} = \mathbf{v}_3 + \mathbf{v}_{17} - \mathbf{v}_6 = \mathbf{v}_6 + \mathbf{v}_{10} + \mathbf{v}_{24} \tag{12}$$

$$\mathbf{v}_{14} = \mathbf{v}_7 + \mathbf{v}_{23} - \mathbf{v}_{16} = \mathbf{v}_4 + \mathbf{v}_{16} + \mathbf{v}_{20} \tag{13}$$

$$\mathbf{v}_{18} = \mathbf{v}_7 + \mathbf{v}_{15} - \mathbf{v}_4 = \mathbf{v}_4 + \mathbf{v}_{16} + \mathbf{v}_{24} \tag{14}$$

$$\mathbf{v}_{18} = \mathbf{v}_9 + \mathbf{v}_{11} - \mathbf{v}_2 = \mathbf{v}_{20} + \mathbf{v}_{22} \tag{15}$$

$$\mathbf{v}_1 = \mathbf{v}_8 + \mathbf{v}_{12} - \mathbf{v}_{19} = \mathbf{v}_{16} + \mathbf{v}_{20}$$

$$\tag{16}$$

$$\mathbf{v}_5 = \mathbf{v}_8 + \mathbf{v}_{14} - \mathbf{v}_{17} = 2\mathbf{v}_{10} + \mathbf{v}_{22}$$

$$\mathbf{v}_{21} = \mathbf{v}_{12} + \mathbf{v}_{18} - \mathbf{v}_9 = \mathbf{v}_4 + 2\mathbf{v}_{16} + \mathbf{v}_{24}$$

$$\mathbf{v}_{25} = \mathbf{v}_{14} + \mathbf{v}_{18} - \mathbf{v}_7 = \mathbf{v}_6 + \mathbf{v}_{10} + 2\mathbf{v}_{24}$$

$$\mathbf{v}_{11} + \mathbf{v}_{15} = \mathbf{v}_8 + \mathbf{v}_{18} \Rightarrow \mathbf{v}_{16} = -\mathbf{v}_{10}$$

$$\mathbf{v}_3 + \mathbf{v}_{23} = \mathbf{v}_{12} + \mathbf{v}_{14} \Rightarrow \mathbf{v}_{24} = \mathbf{0}.$$

Equating equations (3),(4) and (9),(10) gives

$$\mathbf{v}_{10} + \mathbf{v}_{22} = \mathbf{v}_4 + \mathbf{v}_6 + \mathbf{v}_{24} \tag{17}$$

$$\mathbf{v}_4 + \mathbf{v}_{16} + \mathbf{v}_{24} = \mathbf{v}_{20} + \mathbf{v}_{22}.$$

Adding the above equations gives $\mathbf{v}_6 + \mathbf{v}_{20} = \mathbf{v}_{10} + \mathbf{v}_{16} = \mathbf{0}$. Equating equations (5),(6) and (7),(8) gives

$$\mathbf{v}_{16} + \mathbf{v}_{20} = \mathbf{v}_6 + \mathbf{v}_{10} + \mathbf{v}_{22} \tag{18}$$

$$\mathbf{v}_6 + \mathbf{v}_{10} + \mathbf{v}_{24} = \mathbf{v}_4 + \mathbf{v}_{16} + \mathbf{v}_{20}.$$

Adding the above equations gives $\mathbf{v}_4 + \mathbf{v}_{22} = \mathbf{v}_{24} = \mathbf{0}$. Substituting into (11) and (12) we obtain:

$$\mathbf{v}_{10} - \mathbf{v}_4 = \mathbf{v}_4 + \mathbf{v}_6$$

$$-\mathbf{v}_{10} - \mathbf{v}_6 = \mathbf{v}_6 + \mathbf{v}_{10} - \mathbf{v}_4.$$

The above system gives $\mathbf{v}_4 = \frac{4}{5}\mathbf{v}_{10}$ and $\mathbf{v}_6 = -\frac{3}{5}\mathbf{v}_{10}$. Now we see that all velocities are scalar multiples of $\mathbf{v}_{10}$. Since $\mathbf{p}_{10}\mathbf{p}_{13}$ is a bar we have that $\mathbf{v}_{10} \cdot (\mathbf{p}_{10} - \mathbf{p}_{13}) = 0$. Since $\mathbf{p}_3\mathbf{p}_{10}$ is a bar we have that $(\mathbf{v}_3 - \mathbf{v}_{10}) \cdot (\mathbf{p}_3 - \mathbf{p}_{10}) = \mathbf{v}_6 \cdot (\mathbf{p}_3 - \mathbf{p}_{10}) = -\frac{3}{5}\mathbf{v}_{10} \cdot (\mathbf{p}_3 - \mathbf{p}_{10}) = 0$. The directions of the bars $\mathbf{p}_{10}\mathbf{p}_{13}$ and $\mathbf{p}_3\mathbf{p}_{10}$ are linearly independent, and so $\mathbf{v}_{10} = \mathbf{0}$. It follows that all velocities are zero and N_5 is infinitesimally rigid. $\square$

The framework obtained by deleting the corner joints and one degree three joint from the 5×5 knight's framework is also infinitesimally rigid, see Figure 10. This framework has 20

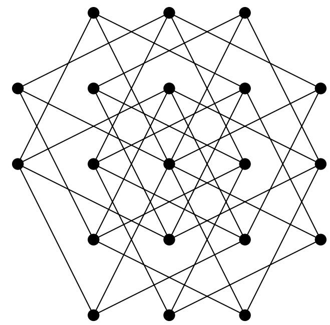


Figure 10: Trimmed 5×5 knight's framework

joints and 37 edges. The rigidity of this framework can be verified using a similar approach to the above, or by calculating the rank of its rigidity matrix.

Note that $F_{m+1,n}(5)$ can be constructed from $F_{m,n}(5)$ by attaching n joints each with at least two non-parallel bars. As a result, $F_{m,n}(5)$ is infinitesimally rigid for all $m, n \geq 5$. In addition, by the same verification techniques, $F_{4,7}(5)$ is also infinitesimally rigid. This implies $F_{4,n}(5)$ and $F_{m,4}(5)$ are infinitesimally rigid for $m, n \geq 7$. This completely describes the infinitesimally rigid $F_{m,n}(5)$, since all other such frameworks have less than $2mn-3$ bars.

2.2 Leaper Frameworks

The knight's graph is one instance of a leaper graph. An $\{r, s\}$-leaper graph on an $m \times n$ board has the vertex set $\{(x, y) \in \mathbb{Z}^2 \colon 0 \leq x \leq m-1, 0 \leq y \leq n-1\}$, and edges between pairs of points of the form (x, y) and $(x \pm r, y \pm s)$ or (x, y) and $(x \pm s, y \pm r)$. Edges represent the legal moves of a generalized knight chess piece that jumps in an L-shape with side lengths r and s. We call the natural embedding of a leaper graph into $\mathbb{R}^2$ an *leaper framework*. See Figure 11 for example. Knuth [6] proved the following theorem on the connectivity of leaper graphs.

Theorem 2.3. (Knuth) The $\{r, s\}$-leaper graph on an $m \times n$ board, where $2 \leq m \leq n$ and $1 \leq r \leq s$, is connected if and only if the folloing three conditions hold: (i) $r + s$ is relatively prime to $r - s$; (ii) $n \geq 2s$; (iii) $m \geq r + s$. □

We have verified that the (r, s)-leaper framework on an $m \times n$ chessboard is infinitesimally

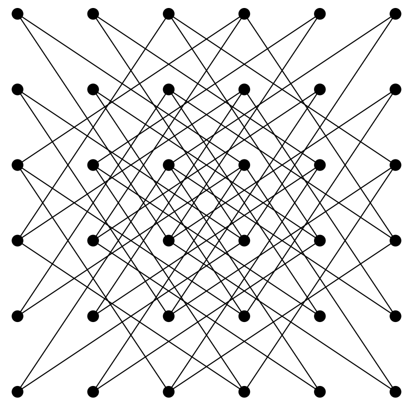


Figure 11: 6×6 $\{2,3\}$-leaper framework

rigid if $r+s$ is relatively prime to $r-s$ and $r, s \geq 2(r+s)-1$, for $1 \leq r, s \leq 25$. We expect that a rigid analogue of Knuth's connected theorem above exists for leaper frameworks. That is, under the right conditions, a large enough leaper framework will be infinitesimally rigid.

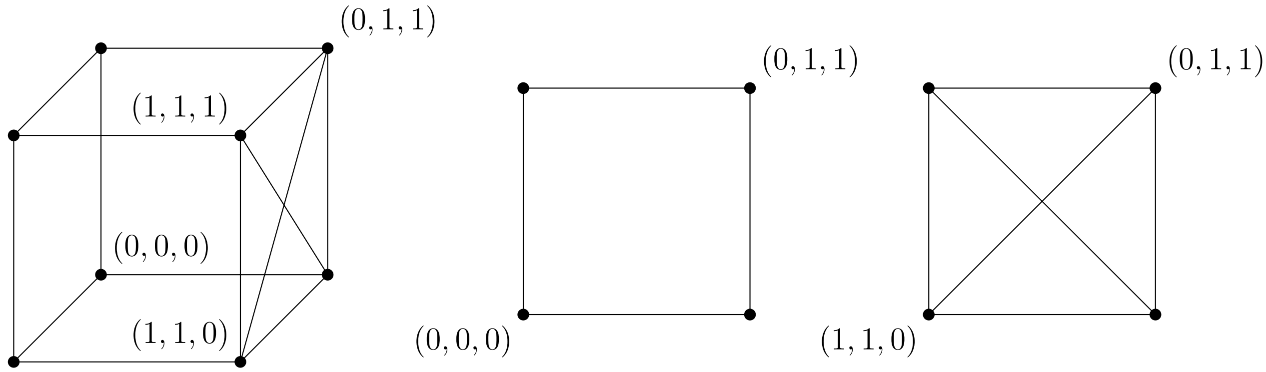
3 Bipartite Unit-Bar Frameworks in $\mathbb{R}^d$

The results of this section correspond with Section 2 in [12]. The knight's framework can be extended to higher dimensions. We do this by constructing the knight's framework on two-dimensional cross-sections within a higher dimensional lattice. We prove the resulting framework is infinitesimally rigid. Furthermore, and of perhaps broader interest, we prove more generally that a framework on an integer lattice in any dimension with infinitesimally rigid cross-sections is infinitesimally rigid.

Definition 3.1. An n-lattice framework in $\mathbb{R}^d$ has joints of the form $(x_1, \ldots, x_d)$, where $x_i \in \{0, 1, \ldots, n-1\}$. Let F be an n-lattice framework in $\mathbb{R}^d$. For all integers $1 \leq i \leq d$ and $0 \leq c \leq n-1$, define $F_{i,c}$ to be the cross-section framework of F induced by all joints in F of the form $(x_1, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_d)$. The framework $F_{i,c}$ can be embedded in $\mathbb{R}^{d-1}$ by deleting the i^{th} coordinate of all joints in $F_{i,c}$. The resulting framework is an n-lattice framework in $\mathbb{R}^{d-1}$, call it $F'_{i,c}$. $\square$

In Figure 12, the cross-section frameworks $F_{1,0}$ and $F_{2,1}$ are infinitesimally flexible in $\mathbb{R}^3$.

In $\mathbb{R}^2$, $F'_{1,0}$ remains infinitesimally flexible, but $F'_{2,1}$ is infinitesimally rigid.



(a) A 2-lattice framework F (b) $F_{1,0}$ (c) $F_{2,1}$

Figure 12: Cross-sections of a lattice framework

The framework F in Figure 12 is infinitesimally flexible. If all 6 faces of F had the two diagonal bars as $F_{2,1}$ does, it's easy to check that the framework would be infinitesimally rigid. We'll prove that in general, if the cross-sections have all bars, then the framework is infinitesimally rigid.

Lemma 3.2. Let $\mathbf{y}, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be joints of a framework F such that $\mathbf{y}\mathbf{x}_i$ is a bar for all i. Let f be an infinitesimal motion of F such that $f(\mathbf{x}_i) = \mathbf{0}$ for all i. If $\mathbf{z}$ is in the span of $\{\mathbf{y} - \mathbf{x}_1, \ldots, \mathbf{y} - \mathbf{x}_n\}$, then $f(\mathbf{y}) \cdot \mathbf{z} = 0$.

Proof. Let $\mathbf{z} = a_1(\mathbf{y} - \mathbf{x}_1) + \cdots + a_n(\mathbf{y} - \mathbf{x}_n)$ with $a_i \in \mathbb{R}$. Since $\mathbf{y}\mathbf{x}_i$ is an edge, $(f(\mathbf{y}) - f(\mathbf{x}_i)) \cdot (\mathbf{y} - \mathbf{x}_i) = f(\mathbf{y}) \cdot (\mathbf{y} - \mathbf{x}_i) = 0$ for all i. Hence

$$f(\mathbf{y}) \cdot \mathbf{z} = a_1 f(\mathbf{y}) \cdot (\mathbf{y} - \mathbf{x}_1) + \cdots + a_n f(\mathbf{y}) \cdot (\mathbf{y} - \mathbf{x}_n) = 0.$$

$\square$

When the dimension is unambiguous, we will use the notation $\mathbf{e}_k$ to represent the standard basis vector consisting of a 1 in the k^{th} entry and zeroes elsewhere. The vector $\mathbf{e}_k$ will represent both the direction, and the joint with the corresponding coordinates. The context will make the use clear.

Theorem 3.3. Let F be an n-lattice framework in $\mathbb{R}^d$, $d \geq 3$, and $n \geq 2$. If for all $1 \leq i \leq d$ and $0 \leq c \leq n-1$, the framework $F_{i,c}$ has bars between all pairs of joints, then F is infinitesimally rigid.

Proof. Consider all infinitesimal motions f of F such that

$$f(\mathbf{0}) = \mathbf{0}, \quad \text{and} \quad f_i(\mathbf{e}_k) = 0 \quad \text{for} \quad 1 \leq k \leq d-1 \quad \text{and} \quad k+1 \leq i \leq d. \tag{19}$$

The restrictions of (19) specify $d+(d-1)+\ldots+1 = \binom{d+1}{2}$ degrees of freedom of f. Hence the space of infinitesimal motions of F that satisfy (19) is at most $\binom{d+1}{2}$ less than the dimension of the space of all infinitesimal motions of F. It follows that if the only infinitesimal motions of F that satisfy (19) are identically zero, then F is infinitesimally rigid.

Let f be an infinitesimal motion of F satisfying (19). Note that $\mathbf{e}_1\mathbf{0}$ is a bar of F and $\mathbf{e}_1$ is in the span of $\{\mathbf{e}_1 - \mathbf{0}\}$. Since $f(\mathbf{0}) = \mathbf{0}$, by Lemma 3.2 we see that $f(\mathbf{e}_1) \cdot \mathbf{e}_1 = \mathbf{0}$ and so $f(\mathbf{e}_1) = \mathbf{0}$. Notice $\mathbf{e}_i\mathbf{0}$ is a bar for all $1 \leq i \leq d$. For all $j \neq i$, since $d \geq 3$, we have that $\mathbf{e}_i\mathbf{e}_j$ is also a bar. A simple induction and Lemma 3.2 gives the result $f(\mathbf{e}_i) = \mathbf{0}$ for all $1 \leq i \leq d$. For any joint $\mathbf{x} \in F_{i,0}$ we have that $\mathbf{x0}$ and $\mathbf{x}\mathbf{e}_j$ are bars for all $j \neq i$. Lemma 3.2 gives $f_j(\mathbf{x}) = 0$ for all $j \neq i$. Hence if a joint $\mathbf{x}$ has a zero in two or more coordinates, $f(\mathbf{x}) = \mathbf{0}$. Let $\mathbf{y} = (y_1, \ldots, y_d)$ be a joint of F such that $y_i \neq 0$ for all i. Let $\mathbf{y}^{(i)}$ be the joint with $\mathbf{y}_i$ in the i^{th} coordinate and zeros in all other coordinates. Notice that $\mathbf{y}\mathbf{y}^{(i)}$ is a bar for all $1 \leq i \leq d$. Furthermore, since $d \geq 3$, $\mathbf{y}^{(i)}$ has a zero in at least two coordinates, and so $f(\mathbf{y}^{(i)}) = \mathbf{0}$. It is easy to check that the span of $\{\mathbf{y} - \mathbf{y}^{(i)}\}_{1 \leq i \leq d}$ is all of $\mathbb{R}^d$, and so by Lemma 3.2, $f(\mathbf{y}) = \mathbf{0}$. Finally, let $\mathbf{x} = (x_1, \ldots, x_d)$ be a joint of F such that $x_i = 0$ and $x_j \neq 0$ for $j \neq i$. Let $\mathbf{z}$ be the joint $(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_d)$. The existence of $\mathbf{z}$ follows from $n \geq 2$. Since $\mathbf{xz}$ is a bar:

$$(f(\mathbf{x}) - f(\mathbf{z})) \cdot (\mathbf{x} - \mathbf{z}) = (f(\mathbf{x}) - f(\mathbf{z})) \cdot \mathbf{e}_i = 0.$$

Since all coordinates of $\mathbf{z}$ are nonzero, $f(\mathbf{z}) = \mathbf{0}$, and so $f_i(\mathbf{x}) = 0$. It follows that $f \equiv \mathbf{0}$, and F is an infinitesimally rigid framework.

□

Note that the assumptions $d \geq 3$ in the above theorem is necessary. In the plane, the cross-sections of the square are single bar frameworks, and are infinitesimally rigid in $\mathbb{R}$, but the square is flexible. Using the same argument, Theorem 3.3 can be extended to a framework on a lattice of the form $[0, a_1] \times [0, a_2] \times \cdots \times [0, a_d]$, where each $a_i \geq 1$. The assumption $a_i \geq 1$ is necessary, since a 'flat' framework will have an infinitesimal motion in the perpendicular direction. For an example, see $F_{2,1}$ in Figure 12.

The assumption in Theorem 3.3 that all bars be present in all cross-sections can be replaced with the cross-sections being infinitesimally rigid.

Corollary 3.4. Let F be an n-lattice framework in $\mathbb{R}^d$, $d \geq 3$ and $n \geq 2$. If for all $1 \leq i \leq d$, and $0 \leq c \leq n-1$, the framework $F'_{i,c}$ is infinitesimally rigid, then F is infinitesimally rigid.

Proof. Let $1 \leq i \leq d$ and $0 \leq c \leq n_i - 1$ be arbitrary. Any infinitesimal motion f of F induces an infinitesimal motion $f_{i,c}$ of $F_{i,c}$ in the following way. For any joint $\mathbf{x} \in F'_{i,c}$ let $\hat{\mathbf{x}}$ denote the corresponding joint in $F_{i,c}$, and put

$$f_{i,c}(\mathbf{x}) = [f_1(\hat{\mathbf{x}}) \ldots f_{i-1}(\hat{\mathbf{x}})\ f_{i+1}(\hat{\mathbf{x}}) \ldots f_d(\hat{\mathbf{x}})]^t\,.$$

It is clear that this defines $f_{i,c}$ as a vector field in $\mathbb{R}^{d-1}$. Furthermore, for any bar $\mathbf{xy}$ of $F'_{i,c}$, since f is an infinitesimal motion:

$$(f_{i,c}(\mathbf{x}) - f_{i,c}(\mathbf{y})) \cdot (\mathbf{x} - \mathbf{y}) = (f(\hat{\mathbf{x}}) - f(\hat{\mathbf{y}})) \cdot (\hat{\mathbf{x}} - \hat{\mathbf{y}}) = 0. \tag{20}$$

It follows that $f_{i,c}$ is an infinitesimal motion. Notice that the first equality in (20) holds for all $\mathbf{x}, \mathbf{y} \in F'_{i,c}$, and not just bars. Since $F'_{i,c}$ is infinitesimally rigid we see that both equalities in (20) holds for all $\mathbf{x}, \mathbf{y} \in F_{i,c}$. Hence all infinitesimal motions of F are infinitesimal motions of the framework described in Theorem 3.3, and so F is infinitesimally rigid.

□

Definition 3.5. The $n \times \cdots \times n$ knight's framework in $\mathbb{R}^d$ is the n-lattice framework with bars between two joints $\mathbf{x}$ and $\mathbf{y}$ if the coordinates of $\mathbf{x}$ and $\mathbf{y}$ are equal except in two places where they differ by 1 and 2. □

All bars in the knight's framework have length $\sqrt{5}$. The parity of the sum of the coordinates of two adjacent joints is different, the same as in the two dimensional case. Hence the knight's framework in $\mathbb{R}^d$ is bipartite, and in particular, triangle free. A consequence of Theorem 2.2 and Corollary 3.4 is the following.

Theorem 3.6. The $5 \times \cdots \times 5$ knight's framework in $\mathbb{R}^d$, for $d \geq 2$, is an infinitesimally rigid bipartite unit-bar framework. $\square$

Using a computer and the rigidity matrix we noticed that the $4 \times 4 \times 4$ knight's framework is infinitesimally rigid. The computer code of this program that does this verification can be found in Appendix 1 of [12]. It follows by Corollary 3.4 that the $4 \times \cdots \times 4$ knight's framework in $\mathbb{R}^d$ for $d \geq 3$ is also infinitesimally rigid.

4 Unit-Bar Frameworks of Higher Girth in $\mathbb{R}^2$

The results of this section correspond with Section 3 in [12]. We extend Maehara's problem of finding triangle-free infinitesimally rigid unit-bar frameworks, to the problem of finding infinitesimally rigid unit-bar frameworks of higher girth. As noted earlier, building infinitesimally rigid frameworks by attaching triangles is easy since triangles are infinitesimally rigid. On the other hand, finding a triangle-free infinitesimally rigid framework is harder, since components of the framework are infinitesimally flexible, i.e. every four joints induce an infinitesimally flexible framework. Along the same reasoning we see that finding an infinitesimally rigid unit-bar framework of girth g is more problematic still, since every set of g joints induce an infinitesimally flexible framework.

Erdős' construction of many unit distances motivated our approach to finding infinitesimally rigid unit-bar frameworks with larger girth. We consider subframeworks of an $n \times n$ lattice of joints with bars of length $\sqrt{m}$, where m can be written as the sum of two squares in several ways. For odd m, two numbers summing to m have different parity. Hence the sum of the coordinates of adjacent joints is different, and the framework is bipartite. One can show that for even m a framework constructed in this way is also bipartite, see for example

[2]. The following algorithm gives an outline of how we construct our frameworks.

4.1 Generating a unit-bar framework of a chosen girth

Below we describe an algorithm for generating a bipartite unit-bar framework in the plane of a specified girth. The framework in the output is not necessarily infinitesimally rigid.

Algorithm:

Input: The size n of the square lattice, an integer m that can be written as the sum of two squares in several ways, and the desired girth $2g$.

Output: A bipartite unit-bar framework with girth at least $2g$.

(1) Determine all ordered pairs of integers (a, b) where $a^2 + b^2 = m$ and either $b > 0$, or $b = 0$ and $a > 0$. These are the bar directions, call the set D.

(2) Add the joints to the framework, they are at the integer coordinates (x, y) with $0 \leq x, y \leq n - 1$.

(3) Select a permutation σ of the joints at random. For each joint $\mathbf{x}$ make a list $D(\mathbf{x}) = D$ of all possible directions of bars.

(4) In the order described by σ visit each joint $\mathbf{x}$ and do the following.

   i. Randomly select an untried bar direction $\mathbf{d}$ from $D(x)$, let $\mathbf{y} = \mathbf{x} + \mathbf{d}$.

   ii. If $\mathbf{y}$ is a joint in the framework then determine all joints within distance $g - 1$ of $\mathbf{x}$ and distance $g - 2$ of $\mathbf{y}$, call these sets $N_{\mathbf{x}}$ and $N_{\mathbf{y}}$.

   iii. If $N_{\mathbf{x}}$ and $N_{\mathbf{y}}$ are disjoint then add the bar $\mathbf{xy}$ to the framework.

   iv. Remove $\mathbf{d}$ from $D(\mathbf{x})$.

(5) Repeat (4) until $D(\mathbf{x})$ is empty for all joints $\mathbf{x}$, this will take $|D|$ loops.

(6) Remove joints with degree less than three. Output the framework. $\square$

Recall that there is an infinite sequence of natural numbers n such that the number of integer solutions to $p^2 + q^2 = n$ is greater than $n^{c/\log\log n}$ [3]. As a result, for any chosen positive integer M, there exists a lattice framework such that the 'potential' degree of most joints is at least M. An infinitesimally rigid framework in the plane must have average degree approximately 4. The high availability of bars, with respect to needing only an average degree of 4, appeared to make infinitesimally rigid frameworks easy to find using the above algorithm. We expect that the algorithm could be used to find infinitesimally rigid frameworks of arbitrarily large girth, given enough computing power.

We implemented the above algorithm with Python, see Appendix 2 of [12]. For each girth, we experimented with a variety of different m and n, and used many random trials, i.e. different permutations of joints to obtain a large sampling of frameworks with the chosen girth. As remarked earlier the frameworks obtained are not necessarily infinitesimally rigid; this needed to be checked.

4.2 Testing for infinitesimal rigidity

We used built-in functions of Python and Matlab to determine the rank of the rigidity matrix of the frameworks obtained by the algorithm previously described. We noticed that if m was chosen such that it could be written as a sum of two squares in sufficiently many ways in relation to the target girth, an infinitesimally rigid framework could be found from a small sample of outputted frameworks.

Table 1 describes the smallest infinitesimally rigid framework of each girth we found among the frameworks generated by our random trials. The column '# of Trials' represents the approximate total number of frameworks of the described specifications that we tested for infinitesimal rigidity. Many of the frameworks tested in the trials were infinitesimally rigid. The purpose of performing many trials was to find frameworks as small as possible.

Note that in the girth four case, only one trial is needed, since all bars are always added. As the girth increased, we needed to increase n in ordered for enough bars to be available to add. Large frameworks take longer for our program to construct, and evaluate the

Girth	Size n	Square bar length m	# of Joints	# of Edges	# of Trials
4	5	$5 = 1^2 + 2^2$	21	40	1
6	9	$5 = 1^2 + 2^2$	54	105	16000000
8	23	$65 = 1^2 + 8^2$ $= 4^2 + 7^2$	433	865	650000
10	53	$1105 = 4^2 + 33^2$ $= 9^2 + 33^2$ $= 12^2 + 31^2$ $= 23^2 + 24^2$	2467	4931	5000
12	147	$5525 = 7^2 + 74^2$ $= 14^2 + 73^2$ $= 22^2 + 71^2$ $= 25^2 + 70^2$ $= 41^2 + 62^2$ $= 50^2 + 55^2$	18924	37845	10

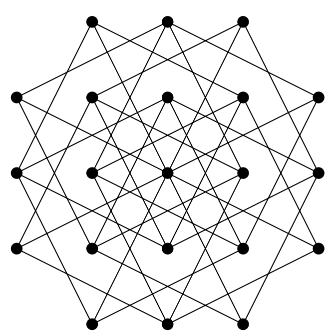
Table 1: Framework specifications

corresponding rank. As a result, because of time constraints, less trials could be performed for frameworks of larger girth. In the case of girth 14, the required computations were too long to perform even one construction.

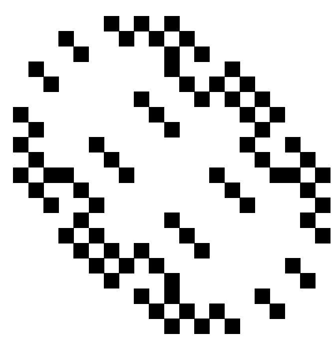
We used Matlab's rank function to double-check the rank calculations of Python for frameworks with girth 4,6,8, and 10. A rank computation for the 37845×37848 rigidity matrix corresponding to the girth 12 framework was impractical for our computers. Our strategy to verify its infinitesimal rigidity was to hand the matrix to Matlab as a sparse matrix. A sparse matrix is a matrix with zero in most entries. A sparse matrix is stored by specifying the locations and values of the nonzero entries, resulting in less used memory. Rank calculations cannot be performed in Matlab on sparse matrices, but singular value calculations can. Recall that the number of nonzero singular values of a matrix is it's rank. The Matlab function 'svds' computes the smallest singular values of matrix. Matlab computed the smallest singular value of the rigidity matrix of our girth 12 framework to be $4.9445 \cdot 10^{-4}$. This value was reproduced upon decreasing the convergence tolerance and increasing the number of iterations of the svd algorithm. This calculation indicates that all singular values of the rigidity matrix are nonzero and the framework is infinitesimally rigid.

4.3 Search Results

Below we draw the frameworks in Table 1 with girth 4,6, and 8. For these frameworks we also record their adjacency matrices below by representing ones with black squares and zeros with white squares. For the frameworks with girth 10 and 12 we record their adjacency matrices using darker shading to represent higher density of edges.

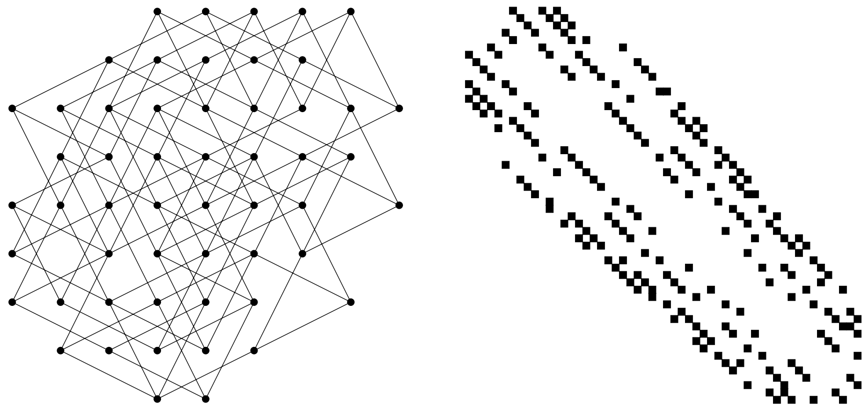


(a) Framework



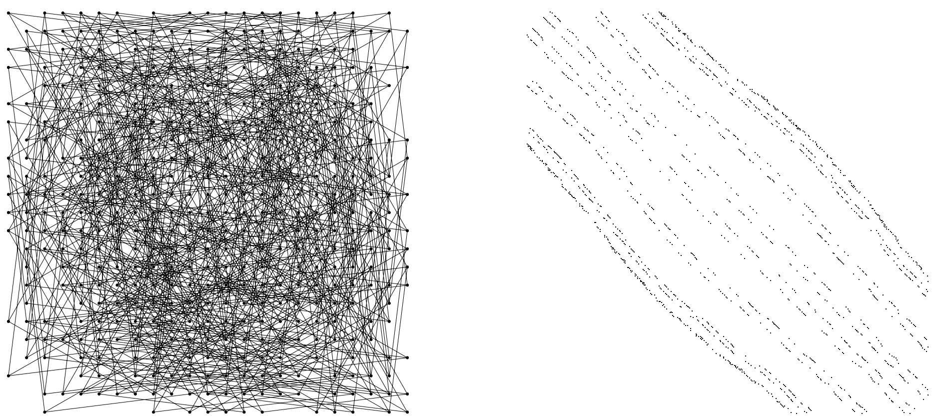
(b) Adjacency matrix

Figure 13: Girth 4



(a) Framework (b) Adjacency matrix

Figure 14: Girth 6



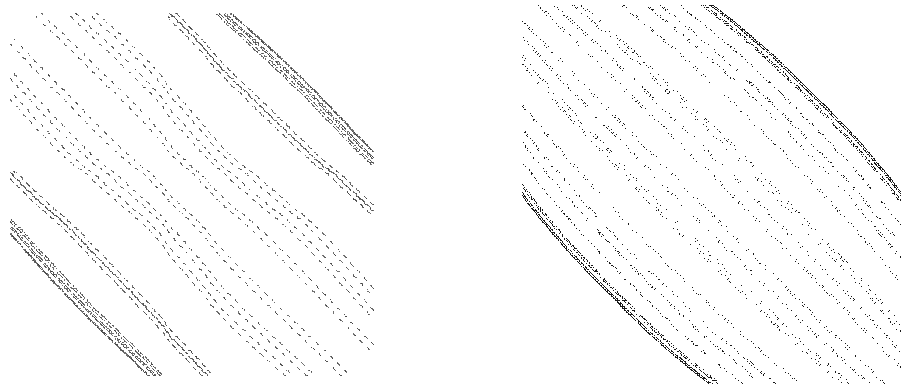
(a) Framework (b) Adjacency matrix

Figure 15: Girth 8

5 Conclusion

5.1 Summary of results

The results of this work are constructions of infinitesimally rigid bipartite unit-bar frameworks. Determining an infinitesimally rigid bipartite framework in the plane and in higher dimensions was a problem asked by Hiroshi Maehara over two decades ago. The knight's framework and our generalization of it to any dimension resolve these problems. One reason rigid bipartite unit-bar frameworks are interesting is that most rigid frameworks we think



(a) Girth 10 (b) Girth 12

Figure 16: Girth 10 and 12

of contain triangles. Triangles are the only rigid polygonal frameworks. Squares, pentagons, hexagons, etc, are all increasingly flexible, that is, they have successively more degrees of freedom. As a result, finding frameworks with higher girth is an interesting extension of the triangle-free frameworks constructed by Maehara. The infinitesimally rigid bipartite unit-bar frameworks of girth 4,6,8,10, and 12 presented in this thesis were found by searching integer lattices in the plane with a single distance determined by many pairs of points, an idea borrowed from Paul Erdős' construction of many unit-distances in the plane.

5.2 Future Work

There are several different avenues that would extend the results of this thesis. Leaper frameworks (defined in Section 2.2) are a generalization of the knight's framework. We have verified for finitely many case that if $r + s$ and $r - s$ are relatively prime, an $\{r, s\}$-leaper framework is infinitesimally rigid on a $[2(r + s) - 1] \times [2(r + s) - 1]$ grid. We expect that this, or a comparable result will be true in general.

Maehara constructed an infinitesimally rigid triangle-free unit-bar framework in the plane with 22 joints. In this thesis we presented an infinitesimally rigid bipartite unit-bar framework in the plane with 20 joints. We are curious as to the number of joints in the smallest infinitesimally rigid unit-bar bipartite framework. More generally, using more random trials we expect that smaller examples of the frameworks of girth 6 and larger in Table 1 can be found. Improvements to the algorithm in Section 4.1 might offer a more efficient way to find smaller examples, and examples of infinitesimally rigid unit-bar frameworks with girth larger than 12.

We are limited by computing power in finding infinitesimally rigid unit-bar frameworks of larger girth, though we expect they exist for arbitrarily large girth. We pose the problem of proving the existence of such frameworks.

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