Critical Substructures

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A foundational result in Extremal Graph Theory is as follows. Let $ex(m, G)$ denote the maximum number of edges in a simple graph on m vertices such that there is no subgraph G.
The Turán graph $T(m, k)$ on m vertices are formed by partitioning m vertices into k nearly equal sets and joining any pair of vertices in different sets.

A foundational result in Extremal Graph Theory is as follows. Let $ex(m, G)$ denote the maximum number of edges in a simple graph on m vertices such that there is no subgraph G.
The Turán graph $T(m, k)$ on m vertices are formed by partitioning m vertices into k nearly equal sets and joining any pair of vertices in different sets.

Theorem (Mantel 07, Turán 41) Let K_k denote the clique on k vertices (every pair of vertices are joined). Then
$ex(m, K_k) = |E(T(m, k-1))|$.

We say $\mathcal{H} = ([m], \mathcal{E})$ is a hypergraph if $\mathcal{E} \subseteq 2^{[m]}$. The subsets in $\mathcal{E}$ are called hyperedges.

Consider a hypergraph $H = ([4], \mathcal{E})$ with vertices $[4] = \{1, 2, 3, 4\}$ and with the following hyperedges :

$$\mathcal{E} = \{\emptyset, \{1,2,4\}, \{1,4\}, \{1,2\}, \{1,2,3\}, \{1,3\}\} \subseteq 2^{[4]}$$

The incidence matrix A of the hyperedges $\mathcal{E} \subseteq 2^{[4]}$ is:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Graphs $\rightarrow$ Hypergraphs $\sim$ Simple Matrices

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Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.

Definition We define $\|A\|$ to be the number of columns in A.

$$\|A\| = 6 = |\mathcal{E}|$$

Definition Given a matrix F, we say that A has F as a *configuration* if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \;\; 0 \;\; 1} & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \;\; 1 \;\; 0} & & 0 & \boxed{0} \end{bmatrix}$$

$ex(m, G) \rightarrow forb(m, F)$

We consider the property of forbidding a configuration F in A.

Definition Let

$forb(m, F) = \max\{\|A\| : A \ m\text{-rowed simple, no configuration } F\}$

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$$\text{e.g. } forb(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = m + 1$$

Some Main Results

Let K_k denote the $k \times 2^k$ simple matrix (all columns on k rows)

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

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Corollary *Let F be a $k \times \ell$ simple matrix. Then $forb(m, F) = O(m^{k-1})$.*

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Theorem (Füredi 83). *Let F be a $k \times \ell$ matrix. Then $forb(m, F) = O(m^k)$.*

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Corollary *Let F be a $k \times \ell$ simple matrix. Then* $\mathit{forb}(m, F) = O(m^{k-1})$.

Theorem (Füredi 83). *Let F be a $k \times \ell$ matrix. Then* $\mathit{forb}(m, F) = O(m^k)$.

Problem Given F, can we predict the behaviour of $\mathit{forb}(m, F)$?

Results for K_4

$$K_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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Theorem (Vapnik and Chervonenkis 71, Perles and Shelah 72, Sauer 72)

$$forb(m, K_4) = \binom{m}{3} + \binom{m}{2} + \binom{m}{1} + \binom{m}{0}.$$

Critical Substructures

We define F' to a critical substructure of F if F' is a configuration in F and

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Note that for F'' which contains F' where F'' is contained in F, we deduce that

$$forb(m, F') = forb(m, F'') = forb(m, F).$$

Critical Substructures for K_3, K_4

The critical substructures for K_3 follows from work of A, Karp 10 while the critical substructures for K_4 follows from work of A, Raggi 11. We need some difficult base cases to establish the critical substructures for K_5.

Miguel Raggi

Steven Karp

Critical Substructures for K_4

$$K_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Critical substructures are $\mathbf{1}_4$, K_4^3, K_4^2, K_4^1, $\mathbf{0}_4$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.
Note that $\mathit{forb}(m, \mathbf{1}_4) = \mathit{forb}(m, K_4^3) = \mathit{forb}(m, K_4^2) = \mathit{forb}(m, K_4^1)$ $= \mathit{forb}(m, \mathbf{0}_4) = \mathit{forb}(m, 2 \cdot \mathbf{1}_3) = \mathit{forb}(m, 2 \cdot \mathbf{0}_3)$.

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Critical Substructures for K_4

$$K_4 = \left[\begin{array}{ccccc|cccccc|ccccc} 1 & 1 & 1 & 1 & 0 & \color{red}1 & \color{red}1 & \color{red}1 & \color{red}0 & \color{red}0 & \color{red}0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & \color{red}1 & \color{red}0 & \color{red}0 & \color{red}1 & \color{red}1 & \color{red}0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & \color{red}0 & \color{red}1 & \color{red}0 & \color{red}1 & \color{red}0 & \color{red}1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & \color{red}0 & \color{red}0 & \color{red}1 & \color{red}0 & \color{red}1 & \color{red}1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Critical substructures are $\mathbf{1}_4$, K_4^3, $\color{red}K_4^2$, K_4^1, $\mathbf{0}_4$, $2 \cdot \mathbf{1}_3$, $2 \cdot \mathbf{0}_3$.
Note that $forb(m, \mathbf{1}_4) = forb(m, K_4^3) = forb(m, K_4^2) = forb(m, K_4^1)$
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$$K_4 = \left[\begin{array}{ccccccccccccccc|c|} 1&1&1&1&0&1&1&1&0&0&0&1&0&0&0&\color{red}{0}\\ 1&1&1&0&1&1&0&0&1&1&0&0&1&0&0&\color{red}{0}\\ 1&1&0&1&1&0&1&0&1&0&1&0&0&1&0&\color{red}{0}\\ 1&0&1&1&1&0&0&1&0&1&1&0&0&0&1&\color{red}{0} \end{array}\right]$$

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$$K_4 = \left[\begin{array}{cc|cccccccccccccc} \color{red}{1} & \color{red}{1} & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \color{red}{1} & \color{red}{1} & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \color{red}{1} & \color{red}{1} & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

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Theorem The k-rowed critical substructures of K_k are K_k^ℓ for $0 \le \ell \le k$.
Conjecture The critical substructures of K_k are K_k^ℓ for $0 \le \ell \le k$ and $2 \cdot \mathbf{1}_{k-1}$ and $2 \cdot \mathbf{0}_{k-1}$.

The problem is in showing
$forb(m, [\mathbf{0}_{k-1}\ 2 \cdot K_{k-1}^1 2 \cdot K_{k-1}^2 \cdots 2 \cdot K_{k-1}^{k-2}\ \ \mathbf{1}_{k-1}]) < forb(m, K_k)$
and for this the problem is 'merely' establishing a base case.

We can extend K_4 and yet have the same bound

Using induction, Connor and I were able to extend the bound of Sauer, Perles and Shelah, Vapnik and Chervonenkis. The base cases of the induction were critical.

Connor Meehan after receiving medal

We can extend K_4 and yet have the same bound

$[K_4|\mathbf{1}_2\mathbf{0}_2] =$

$$\left[\begin{array}{cccccccccccccccc|c}
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]$$

Theorem (A., Meehan 10) For $m \geq 5$, we have
$forb(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = forb(m, K_4)$.

We can extend K_4 and yet have the same bound

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Theorem (A., Meehan 10) For $m \geq 5$, we have $forb(m, [K_4|\mathbf{1}_2\mathbf{0}_2]) = forb(m, K_4)$.

We expect in fact that we could add many copies of the column $\mathbf{1}_2\mathbf{0}_2$ and obtain the same bound, albeit for larger values of m.

We can extend K_4 and yet have the same bound

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Are these critical superstructures?

We proved a number of results where
$forb(m, [K_k \mid F]) = forb(m, K_k)$
and also where
$forb(m, [2 \cdot K_k \mid F']) = forb(m, 2 \cdot K_k)$.

We have a number of examples of critical substructures which builds our intuition on how the bounds are determined.

$$F = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

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$$F = \left[\begin{array}{|cc|ccc}\hline \color{red}{1} & \color{red}{1} & 1 & 0 & 0 \\ \color{red}{1} & \color{red}{1} & 0 & 1 & 0 \\ \color{red}{0} & \color{red}{0} & 1 & 0 & 0 \\ \hline\end{array}\right]$$

Critical Substructure

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$$F = \left[\begin{array}{|cc|ccc}\hline 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline\end{array}\right]$$

Critical Substructure

Theorem (A, Karp 10)

$$forb(m, F) \leq \binom{m}{2} + \binom{m}{1} + \binom{m}{0} + \binom{m}{m}.$$

The unique construction is $\binom{[m]}{2} \cup \binom{[m]}{1} \cup \binom{[m]}{0} \cup \binom{[m]}{m}$

We have a number of examples of critical substructures which builds our intuition on how the bounds are determined.

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have a number of examples of critical substructures which builds our intuition on how the bounds are determined.

$$F = \begin{bmatrix} \boxed{\begin{matrix} \color{red}{1} & \color{red}{1} \\ \color{red}{1} & \color{red}{1} \end{matrix}} & \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 1 & 0 \end{matrix} & \begin{matrix} 1 & 0 \end{matrix} \\ \boxed{\begin{matrix} \color{red}{0} & \color{red}{0} \end{matrix}} & \begin{matrix} 0 & 0 \end{matrix} \end{bmatrix}$$

Critical Substructure

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$$F = \begin{bmatrix} 1 & 1 & \boxed{\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}} \\ 1 & 1 & \\ 1 & 0 & 1 & 0 \\ 0 & 0 & \boxed{\begin{matrix} 0 & 0 \end{matrix}} \end{bmatrix}$$

Critical Substructure

We have a number of examples of critical substructures which builds our intuition on how the bounds are determined.

$$F = \begin{bmatrix} 1 & 1 & \boxed{\color{red}{1} \quad \color{red}{1}} \\ 1 & 1 & \color{red}{0} \quad \color{red}{0} \\ 1 & 0 & 1 \quad 0 \\ 0 & 0 & \boxed{\color{red}{0} \quad \color{red}{0}} \end{bmatrix}$$

Critical Substructure

Theorem (A, Karp 10)

$$forb(m, F) \leq \binom{m}{2} + \binom{m}{1} + \binom{m}{0} + \binom{m}{m}.$$

Two constructions are $\binom{[m]}{2} \cup \binom{[m]}{1} \cup \binom{[m]}{0} \cup \binom{[m]}{m}$ and $\binom{[m]}{0} \cup \binom{[m]}{m-2} \cup \binom{[m]}{m-1} \cup \binom{[m]}{m}$

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Critical Substructure

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$$F = \left[\begin{array}{ccc} \boxed{\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}} \\ 1 & 0 & 0 \end{array}\right]$$

Critical Substructure

Theorem (A, Karp 10)

$$forb(m, F) \leq \frac{4}{3}\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$$

with equality for $m \equiv 1, 3 (\text{mod } 6)$.

For $m \equiv 1, 3 (\text{mod } 6)$, we can find a triple system on m points with the property that for every pair i, j, there is precisely one triple containing i, j

There is an easy bound when forbidding a single column.

Theorem

$$forb(m, \mathbf{1}_k\mathbf{0}_\ell) = \sum_{i=0}^{k-1} \binom{m}{i} + \sum_{i=m-\ell+1}^{m} \binom{m}{i}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Critical Substructures

$$\begin{bmatrix} \boxed{\begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{matrix}} & \begin{matrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{matrix} \end{bmatrix}$$

Is this a Critical Substructure?

$$F = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 1 \\ 0 \\ 0 \end{matrix}} & \begin{matrix} 1 \\ 0 \\ 1 \\ 0 \end{matrix} \end{bmatrix}$$

This is not a Critical Substructure

$$F = \left[\begin{array}{|c|c}\hline 1 & 1\\ 1 & 0\\ 0 & 1\\ 0 & 0\\ \hline\end{array}\right]$$

This is not a Critical Substructure

Theorem $\mathit{forb}(m,F) = 4m-4$ while $\mathit{forb}(m,\mathbf{1}_2\mathbf{0}_2) = 2m+2$.

$$F_{a,b,c,d} = \begin{matrix} a\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. \\ b\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. \\ c\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. \\ d\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. \end{matrix} \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

A, Keevash 06 determine the asymptotics of $F_{a,b,c,d}$. The proof is unexpectedly hard and the constants are large. We should be able to do much better.

Columns of $F_{a,b,c,d}$ are $\mathbf{1}_{a+b}\mathbf{0}_{c+d}$ and $\mathbf{1}_a\mathbf{0}_b\mathbf{1}_c\mathbf{0}_d$. If a, b are relatively large compared with c, d, it would seem that $\mathbf{1}_{a+b}\mathbf{0}_{c+d}$ is a critical substructure of $F_{a,b,c,d}$.

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Theorem (A, Karp 10) Let a, b, c, d be given with $a \geq d$ and $b \geq c$.

$forb(m, F_{a,b,c,d}) = forb(m, \mathbf{1}_{a+b}\mathbf{0}_{c+d})$ i.e. $\mathbf{1}_{a+b}\mathbf{0}_{c+d}$ is a critical substructure of $F_{a,b,c,d}$.

for $(c, d) = (1, 0)$ and $a \geq 1$ and $b \geq 2$ or $a = 0$ and $b \geq 3$

for $(c, d) = (0, 1)$ and $a \geq 1$ and $b \geq 1$

for $(c, d) = (1, 1)$ and $a \geq 1$ and $b \geq 2$

Problem Give some conditions on a, b, c, d so that $forb(m, F_{a,b,c,d}) = forb(m, \mathbf{1}_{a+b}\mathbf{0}_{c+d})$.

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