Forbidden Submatrices

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Acknowledgements

This represents joint work with Richard Anstee and Attila Sali

The problem

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- Given a $k \times \ell$ $(0,1)$-matrix F
- Let Avoids(m, F) denote the set of all simple m-row matrices A with no submatrix F (note: row/column order matters)
- Let $\|A\|$ denote the number of columns of A
- We wish to bound $\|A\|$ in terms of m, for $A \in$ Avoids(m, F)

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Definition

Let F be a $k \times \ell$ $(0,1)$-matrix and $m \in \mathbb{N}$. Define

$$\text{fs}(m, F) = \max\{\|A\| : A \in \text{Avoids}(m, F)\}.$$

The conjecture

Conjecture (Anstee, Frankl, Füredi, Pach 1986)

Let F be a $k \times \ell$ $(0,1)$-matrix. Then

$$\mathrm{fs}(m, F) \in O(m^k).$$

That is, there exists a constant c_F such that $\mathrm{fs}(m, F) \leq c_F m^k$.

Known results

Theorem (Anstee, Füredi 1986)

Let F be a $k \times 1$ $(0,1)$-column. Then

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Let F be a $k \times \ell$ $(0,1)$-matrix. Then with $\varepsilon = (k-1)/(13 \log_2 \ell)$,

$$\mathrm{fs}(m, F) \in O(m^{2k-1-\varepsilon}).$$

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Theorem (Anstee, Füredi 1986)

Let F be a $1 \times \ell$ $(0,1)$-row. Then

$$\text{fs}(m, F) \in O(m).$$

New results

Theorem

Let F be a $k \times \ell$ $(0,1)$-matrix with $\ell \geq 2$ and not all columns identical. Then

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Theorem

Let F be one of the following two $2 \times \ell$ $(0,1)$-matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 1 & 0 & \cdots \end{bmatrix} \quad \textit{or} \quad \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \end{bmatrix}.$$

Then

$$\mathrm{fs}(m, F) \in O(m^2).$$

The algorithm

- Scan the matrix A from left to right

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Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix} \qquad F = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

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- If A has no submatrix F, then # contributions $\leq (\ell - 1)\binom{m}{k} \in O(m^k)$ (pigeonhole)

Theorem (Anstee 2000)

Let F *be the* $1 \times \ell$ $(0,1)$*-row*

$$F = [1 \quad 0 \quad 1 \quad 0 \quad \ldots].$$

Then

$$\mathrm{fs}(m, F) \leq (\ell - 1)m.$$

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- But if A has no F, then $\#$ contributions $\leq (\ell - 1)m$ (pigeonhole)
- So $\text{fs}(m, F) \leq (\ell - 1)m$
- General single-row F can be found in an F of the above form

Amortization idea

- We know that # contributions $\in O(m^k)$, but the conjecture says $\|A\| \in O(m^k)$
- In previous proof, we got lucky: every column made a contribution
- In general, this might only work "on average"
  - Columns may make extra contributions that can be saved up and counted towards future columns that themselves fail to contribute (credit)

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  - Columns may make extra contributions that can be saved up and counted towards future columns that themselves fail to contribute (credit)
  - Columns that fail to contribute may need future payment (debt)
  - Keep credit/debt in an account whose size is "small", so that we know "most" columns still make some contribution

Theorem

Let F be the $2 \times \ell$ $(0,1)$-matrix

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix}.$$

Then

$$\mathrm{fs}(m, F) \leq (\ell - 1)\binom{m}{2} + m + 1.$$

Theorem

Let F be the $2 \times \ell$ $(0,1)$*-matrix*

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- Keep an account consisting of all possible fillers that might occur next (credit)

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- Fillers which do occur consume stored credit

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Claims:

1. $|\mathrm{account}| \le m + 1$.

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Claims:

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2. New credits (i.e. new fillers) come from excess contributions.

$$\|A\| \le \#\ \text{contributions} + m + 1 \le (\ell - 1)\binom{m}{2} + m + 1.$$

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (\ell \text{ columns})$$

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   - Initially, the account is

$$\left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

   because the next column of F is $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right]$ on every pair of rows.

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   because the next column of F is $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ on every pair of rows.
   - Introduce a digraph on the row numbers $1, 2, \ldots, m$ where

$$i \longrightarrow j$$

   if the next column of F is ${}^{i}_{j}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ on rows i, j.

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (\ell \text{ columns})$$

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Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix}$$

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Example

$$A = \begin{bmatrix} 0 & \color{red}{1} & 0 & 0 & 1 & 1 \\ 1 & \color{red}{1} & 0 & 1 & 0 & 0 \\ 0 & \color{red}{0} & 1 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix}$$

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$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (\ell \text{ columns})$$

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   - Claim: the digraph is always a total order

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (\ell \text{ columns})$$

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   - Initially, all edges point down, i.e. the account is

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   - After processing a contributing column, "reverse shuffle" rows with 0's and 1's to make the column look like one of the above

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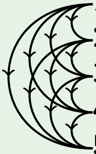
Example



$$\begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \end{matrix} \begin{bmatrix} \cdots & 0 & \cdots \\ \cdots & 1 & \cdots \\ \cdots & 1 & \cdots \\ \cdots & 0 & \cdots \\ \cdots & 1 & \cdots \end{bmatrix}$$

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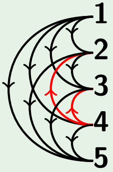
Example



$$\begin{bmatrix} \cdots & 0 & \cdots \\ \cdots & 1 & \cdots \\ \cdots & 1 & \cdots \\ \cdots & 0 & \cdots \\ \cdots & 1 & \cdots \end{bmatrix}$$

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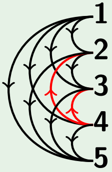

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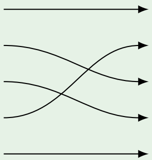

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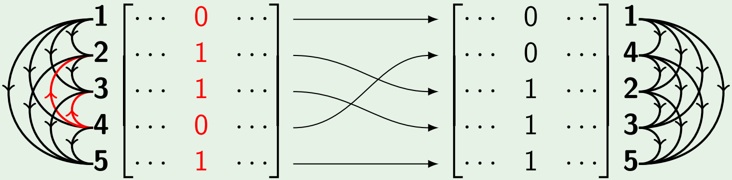

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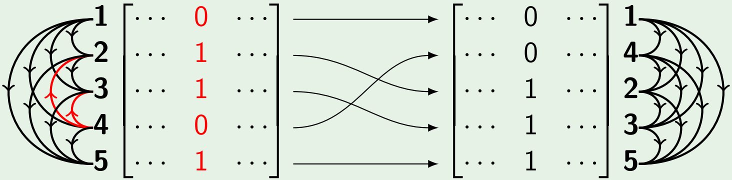

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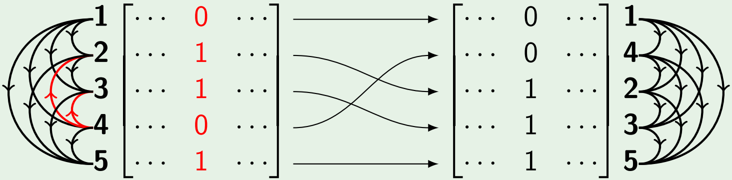

- Flipped edges are unflipped by the shuffling, resulting in a total order
- Filler columns always look the same (0's above 1's) $\Rightarrow$ at most $m + 1$

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (\ell \text{ columns})$$

2. New credits (i.e. new fillers) come from excess contributions.

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   - After the row shuffling, every filler column looks like

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   - After the row shuffling, every filler column looks like

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline 1 \\ \vdots \\ 1 \end{bmatrix}$$

   - Depending on location of split between 0's and 1's, the column might have looked the same *before* the shuffling $\Rightarrow$ not new

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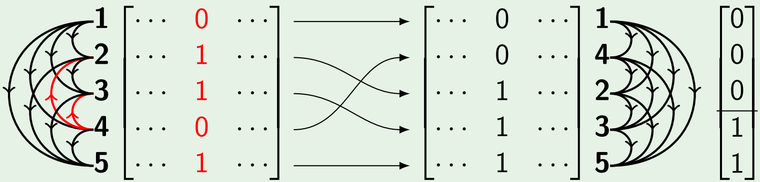

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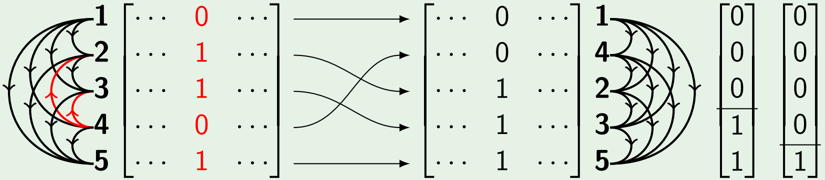

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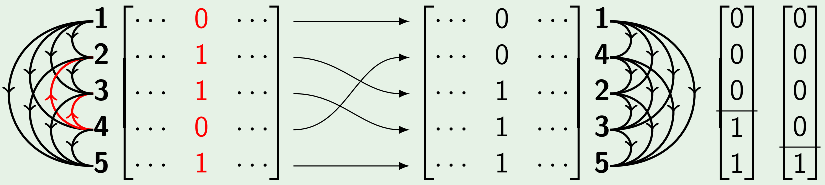

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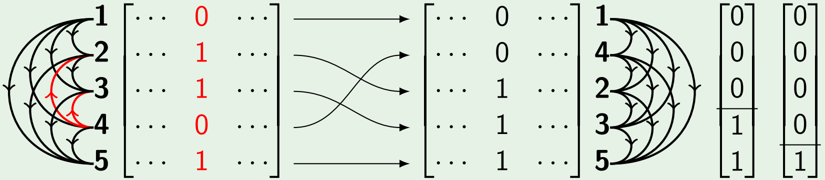

- $\leq 4 - 2$ new fillers, minus the current column
- $\geq 4 - 2$ contributions, minus one for the current column

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2. New credits (i.e. new fillers) come from excess contributions.

$$\|A\| \leq \# \text{ contributions} + m + 1 \leq (\ell - 1)\binom{m}{2} + m + 1.$$

Theorem

Let F *be the* $2 \times \ell$ $(0,1)$*-matrix*

$$F = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 1 & 0 & \cdots \end{bmatrix}.$$

Then

$$\mathrm{fs}(m, F) \leq 2(\ell - 1)\binom{m}{2} + m + 1.$$

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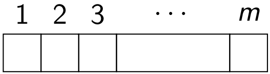

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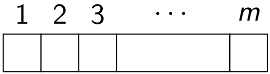

- A contribution on some pair of rows pays for and deletes (up to 2) columns attached to either of those rows

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  - Ignore the column of all 1's (this is our freebie)

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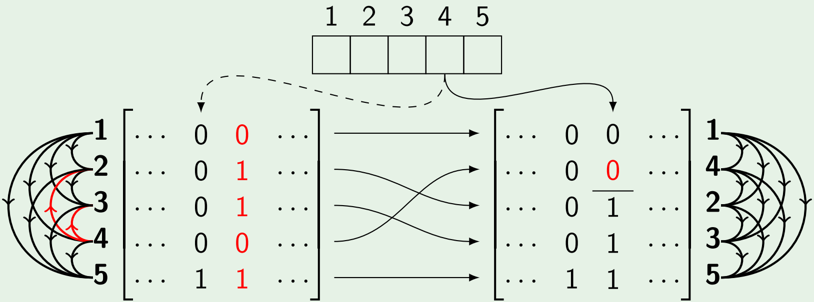
Example



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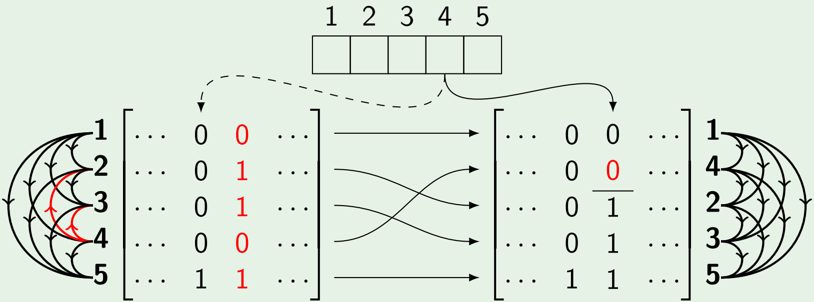

- Any previous column attached to the same row is paid for and deleted

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{bmatrix} \quad (\ell \text{ columns})$$

1. Every column not all 1's is attached to a row.
2. No row is ever attached to more than one column.

$$\|A\| \le 2 \cdot \# \text{ contributions} + m + 1 \le 2(\ell - 1)\binom{m}{2} + m + 1.$$

- Note: we have an extra coefficient of 2 compared to the previous analysis

$$F = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 1 & 0 & \cdots \end{bmatrix} \quad (\ell \text{ columns})$$

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- Proof: complicated