Forbidden Configurations

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- At most how many edges can a simple graph with p vertices have, if it has no triangles?
  Answer: $\left\lfloor \frac{p^2}{4} \right\rfloor$ (Turán's Theorem)

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An m-rowed simple matrix has at most 2^m columns.

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Extremal Problem: If a simple matrix A has m rows and does not have the configuration F, at most how many columns can A have?

Answer: $\text{forb}(m, F)$

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Equivalently, $\text{forb}(m, F)$ is the least integer such that every simple matrix with m rows and more than $\text{forb}(m, F)$ columns has the configuration F.

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What happens if we keep adding $\begin{bmatrix} 1 & 0 \end{bmatrix}$ on top?

Theorem. For $m \geq 3$,

$$\text{forb}(m, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}) = \binom{m}{2} + m + 2.$$

A construction: the m-rowed matrix with all columns of sum

$0, 1, 2$ and m.

Theorem. For $m \geq 4$,

$$\text{forb}(m, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}) = \binom{m}{3} + \binom{m}{2} + m + 2.$$

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Theorem. For $m \geq 5$,

$$\text{forb}(m, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}) = \binom{m}{4} + \binom{m}{3} + \binom{m}{2} + m + 2.$$

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Theorem. For $m \geq k-1 \geq 3$,

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I asked, "What if I flip some digits in the second column?"

Theorem. For $m \geq k-1 \geq 3$,

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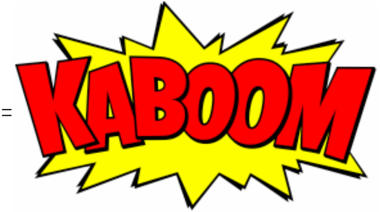

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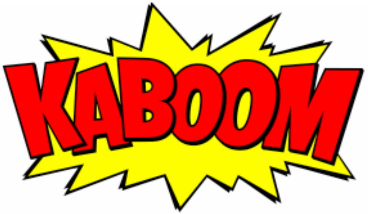

Finding a good construction becomes a difficult Design Theory problem.

What if we flip the 1 at the bottom of the second column to a 0?

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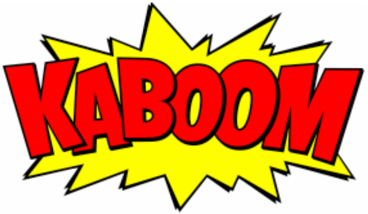

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If we can get a good upper bound on $\#\text{col's}(D)$, then we can prove an upper bound on $\text{forb}(m, F)$ by induction.

Thank You!

Thanks for your attention!