An Introduction to Forbidden Configurations

Connor Meehan
UBC, Vancouver

July 9, 2010
Canadian Undergraduate Mathematics Conference
University of Waterloo

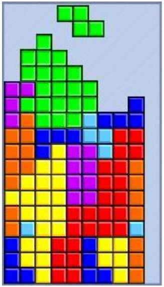
Acknowledgements

My research supervisor Dr. Richard Anstee and associate Miguel Raggi have been a vital part of all my work in this subject area and in preparing this presentation. Thanks go to NSERC for supporting my research with a USRA.

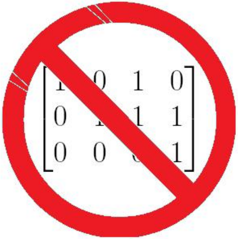
For more information on forbidden configurations, see Dr. Anstee's survey at www.math.ubc.ca/∼anstee.

What Are Forbidden Configurations?

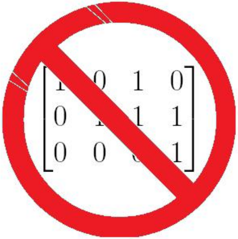
What Are Forbidden Configurations?



What Are Forbidden Configurations?



What Are Forbidden Configurations?



Forbidden configurations are a type of problem in *extremal set theory*. In general, the study of extremal set theory asks the question, “Given a set, what is the largest family of subsets of this set one can attain such that *some property* holds?”

Some definitions make formalizing this idea easier...

Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.

Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.
i.e. if A is $m \times n$, then it is the incidence matrix of some family $\mathcal{A}$ of n subsets of $[m] = \{1, 2, \ldots, m\}$. For example,

$$A = \begin{bmatrix} 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathcal{A} = \{\emptyset, \{2\}, \{3\}, \{1, 3\}, \{1, 2, 3\}\}$$

Each column is a subset of {1,2,3}.

An Easy Extremal Set Problem

An example of an (non-forbidden-configuration) extremal set problem:
What is the largest number of subsets of $\{1,2,3,4\}$ one can have such that each pair of subsets has a non-empty intersection?

An Easy Extremal Set Problem

An example of an (non-forbidden-configuration) extremal set problem:

What is the largest number of subsets of $\{1,2,3,4\}$ one can have such that each pair of subsets has a non-empty intersection?

One could select all subsets that include the element 1:

$$\begin{bmatrix} \boxed{1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1} \\ 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \\ 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \\ 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \end{bmatrix}$$

Each pair of columns intersects along the first row. Thus, the answer is *at least* 8.

An Easy Extremal Set Problem

On the other hand, we can also line up each subset with its complement:

An Easy Extremal Set Problem

On the other hand, we can also line up each subset with its complement:

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \begin{bmatrix}1\\1\\1\\1\end{bmatrix},$$

An Easy Extremal Set Problem

On the other hand, we can also line up each subset with its complement:

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\ \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \qquad \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\ \begin{bmatrix}0\\1\\1\\1\end{bmatrix},$$

An Easy Extremal Set Problem

On the other hand, we can also line up each subset with its complement:

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\ \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\ \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}1\\0\\0\\1\end{bmatrix}\ \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \cdots$$

An Easy Extremal Set Problem

On the other hand, we can also line up each subset with its complement:

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\ \begin{bmatrix}1\\1\\1\\1\end{bmatrix},\qquad \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\ \begin{bmatrix}0\\1\\1\\1\end{bmatrix},\qquad \begin{bmatrix}1\\0\\0\\1\end{bmatrix}\ \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\ \ldots$$

We can only select one subset from each pair, since each pair has an empty intersection. Thus, since there are 8 pairs, the answer is *at most* 8.

Definition Given a matrix F, we say that A has F as a *configuration* if there is a submatrix of A that is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & 0 & \boxed{0} \end{bmatrix} = A$$

Definition Given a matrix F, we say that A has F as a *configuration* if there is a submatrix of A that is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \in \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & 0 & \boxed{0} \end{bmatrix} = A$$

We consider the property of forbidding a configuration F in A for which we say that F is a *forbidden configuration* in A.

Definition Let $\text{forb}(m, F)$ be the largest number of columns that a simple m-rowed matrix A can have subject to the condition that A contains no configuration F. Thus, any $m \times (\text{forb}(m, F) + 1)$ simple matrix contains F as a configuration.

An Easy Forbidden Configuration Problem

What is $\text{forb}\left(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$?

An Easy Forbidden Configuration Problem

What is $\text{forb}\left(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$?

Note that this says that for every pair of columns, one is a subset of the other; otherwise, that pair contains the forbidden configuration.

Thus, we can have only one column of each column sum from 0 to m, and thus at most $m+1$ columns.

An Easy Forbidden Configuration Problem

What is $\text{forb}\left(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$?

Note that this says that for every pair of columns, one is a subset of the other; otherwise, that pair contains the forbidden configuration.

Thus, we can have only one column of each column sum from 0 to m, and thus at most $m+1$ columns.

For example, $m\left\{\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}\right.$.

So $\text{forb}\left(m, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = m+1$.

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows.

e.g. $K_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows.

e.g. $K_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$

Theorem (Sauer 1972, Perles and Shelah 1972, Vapnik and Chervonenkis 1971)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} = \Theta(m^{k-1})$$

Definition Let K_k denote the $k \times 2^k$ simple matrix of all possible columns on k rows.

e.g. $K_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$

Theorem (Sauer 1972, Perles and Shelah 1972, Vapnik and Chervonenkis 1971)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} = \Theta(m^{k-1})$$

- Norbert Sauer: Graph theorist from University of Calgary
- Saharon Shelah: Famous mathematical logician
- Vapnik and Chervonenkis paper was a fundamental one of applied probability

Let $[A|B]$ represent the concatenation of matrices A and B.
Definition Let $q \cdot M$ be the matrix $[M|M|\cdots|M]$ consisting of q copies of M placed side by side.

Theorem (Gronau 1980)

$$forb(m, 2 \cdot K_k) = forb(m, K_{k+1}) = \binom{m}{k} + \binom{m}{k-1} + \cdots + \binom{m}{0}.$$

Where I Come in

My research this summer has looked at extending the applicability of this fundamental result of forbidden configurations. Specifically, the question I've been answering is

What k-rowed matrices G and H are there such that

$$\text{forb}(m, [K_k|G]) = \text{forb}(m, K_k)$$
$$\text{forb}(m, [2 \cdot K_k|H]) = \text{forb}(m, 2 \cdot K_k)?$$

Where I Come in

My research this summer has looked at extending the applicability of this fundamental result of forbidden configurations. Specifically, the question I've been answering is

What k-rowed matrices G and H are there such that

$$\text{forb}(m, [K_k|G]) = \text{forb}(m, K_k)$$
$$\text{forb}(m, [2 \cdot K_k|H]) = \text{forb}(m, 2 \cdot K_k)?$$

An important fact pertaining to this is that if F' is a configuration of F, then $\text{forb}(m, F) \geq \text{forb}(m, F')$ since all matrices that avoid F' necessarily avoid F.

The Standard Induction

By far the most important tool in my research has been induction, the most common manifestation of which uses the *standard decomposition.*

Let A be an $m \times \text{forb}(m, F)$ simple matrix containing no F. We write A as follows upon permuting its columns:

$$A = \begin{bmatrix} 0\,0 \cdots 0\,0 & 1\,1 \cdots 1\,1 \\ B \quad C & C \quad D \end{bmatrix},$$

where C is the matrix of columns that repeat after the first row of A is deleted.

The Standard Induction

By far the most important tool in my research has been induction, the most common manifestation of which uses the *standard decomposition.*

Let A be an $m \times \text{forb}(m, F)$ simple matrix containing no F. We write A as follows upon permuting its columns:

$$A = \begin{bmatrix} 0\,0 \cdots 0\,0 & 1\,1 \cdots 1\,1 \\ B \quad C & C \quad D \end{bmatrix},$$

where C is the matrix of columns that repeat after the first row of A is deleted.

Similarly, if F is k-rowed, we can decompose F after swapping row 1 and row r for all $r \in \{1, ..., k\}$:

$$F = \begin{bmatrix} 0\,0 \cdots 0\,0 & 1\,1 \cdots 1\,1 \\ E_r \quad G_r & G_r \quad H_r \end{bmatrix} \begin{matrix} \leftarrow \text{row } r \\ \\ \end{matrix}.$$

The Standard Induction

$$A = \begin{bmatrix} 0\,0 \cdots 0\,0 & 1\,1 \cdots 1\,1 \\ B \quad C & C \quad D \end{bmatrix} \; F = \begin{bmatrix} 0\,0 \cdots 0\,0 & 1\,1 \cdots 1\,1 \\ E_r \quad G_r & G_r \quad H_r \end{bmatrix} \leftarrow \text{row } r$$

As a specific example, suppose A has no K_3. Then C can have no K_2, as shown:

$$K_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \boxed{\begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{matrix}} & & & & \boxed{\begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{matrix}} \end{bmatrix}$$

The Standard Induction

$$A = \begin{bmatrix} 0\,0 \cdots 0\,0 & 1\,1 \cdots 1\,1 \\ B \quad C & C \quad D \end{bmatrix} \; F = \begin{bmatrix} 0\,0 \cdots 0\,0 & 1\,1 \cdots 1\,1 \\ E_r \quad G_r & G_r \quad H_r \end{bmatrix} \leftarrow \text{row } r$$

In general, we observe that $[BCD]$ is a simple $(m-1)$-rowed matrix that avoids F and C is a simple $(m-1)$-rowed matrix that avoids $[E_r G_r H_r]$ for all $r \in \{1, \ldots, k\}$. Let $|A|$ represent the number of columns in A.

The Standard Induction

$$A = \begin{bmatrix} 0\,0\cdots0\,0 & 1\,1\cdots1\,1 \\ B \quad C & C \quad D \end{bmatrix} \quad F = \begin{bmatrix} 0\,0\cdots0\,0 & 1\,1\cdots1\,1 \\ E_r \quad G_r & G_r \quad H_r \end{bmatrix} \leftarrow \text{row } r$$

In general, we observe that $[BCD]$ is a simple $(m-1)$-rowed matrix that avoids F and C is a simple $(m-1)$-rowed matrix that avoids $[E_r G_r H_r]$ for all $r \in \{1, ..., k\}$. Let $|A|$ represent the number of columns in A.

Thus, if we have induction hypotheses for $\text{forb}(m-1, F)$ and $\text{forb}(m-1, \{[E_r G_r H_r] : r \in \{1, 2, \dots, k\}\})$ that are consistent with base cases, we obtain an upper bound for $\text{forb}(m, F)$ since

$$\text{forb}(m, F) = |A| = |[BCD]| + |C|$$
$$\leq \text{forb}(m-1, F) + \text{forb}(m-1, \{[E_r G_r H_r] : r \in \{1, 2, \dots, k\}\}).$$

Extending K_k by a column

The following theorem is a result of repeated uses of the standard induction and verification of base cases via proof by contradiction.
Theorem Let $k \geq 4$ be a given integer. Let α be a $k \times 1$ (0,1)-column consisting of at least two 1s and at least two 0s. For $m \geq k+1$,

$$\text{forb}(m, [K_k|\alpha]) = \text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}.$$

Extending K_k by a column

The following theorem is a result of repeated uses of the standard induction and verification of base cases via proof by contradiction.

Theorem Let $k \geq 4$ be a given integer. Let α be a $k \times 1$ (0,1)-column consisting of at least two 1s and at least two 0s. For $m \geq k+1$,

$$\text{forb}(m, [K_k|\alpha]) = \text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}.$$

Notice that if α contained at least $k-1$ 1s or 0s, $[K_k|\alpha]$ would contain a $3 \times (k-1)$ matrix of 1s or 0s. Let B be either one of these matrices. It can be shown that $\text{forb}(m, B) > \text{forb}(m, K_k)$ and thus the theorem would no longer be true.

Extending K_k by a column

Theorem Let $q \geq 2$ be a given integer. Then there exists an integer m_0 so that for $m \geq m_0$,

$$\text{forb}(m, \left[K_4 | q \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right]) = \text{forb}(m, K_4) + c_q,$$

where c_q is a constant that depends only on the choice of q.

Extending K_k by a column

Theorem Let $q \geq 2$ be a given integer. Then there exists an integer m_0 so that for $m \geq m_0$,

$$\text{forb}(m, \left[K_4 | q \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right]) = \text{forb}(m, K_4) + c_q,$$

where c_q is a constant that depends only on the choice of q.
It is possible that there exists some m_1 such that for $m \geq m_1$, $\text{forb}(m, \left[K_4 | q \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right]) = \text{forb}(m, K_4)$, but the existence of such a number is as yet unproven.

Sometimes, even if it is known that $\text{forb}(m, [K_k|G]) > \text{forb}(m, K_k)$, it is unclear how to construct best possible extremal matrices. Thus, constructions are sought after.

$$\text{forb}(m, \left[K_2|q \cdot \begin{bmatrix}1\\0\end{bmatrix}\right]) \geq \text{forb}(m, K_2) + \binom{q-2}{2}.$$

(Anstee and Karp, 2008)

Sometimes, even if it is known that $\text{forb}(m, [K_k|G]) > \text{forb}(m, K_k)$, it is unclear how to construct best possible extremal matrices. Thus, constructions are sought after.

$$\text{forb}(m, \left[K_2|q \cdot \begin{bmatrix}1\\0\end{bmatrix}\right]) \geq \text{forb}(m, K_2) + \binom{q-2}{2}.$$

(Anstee and Karp, 2008)

- Steven Karp: Dr. Anstee's 2008 USRA student and student of University of Waterloo!

Extending $2 \cdot K_k$ by a column

Theorem Let $k \geq 2$ be a given integer and let $H = \begin{bmatrix} 1\,1 \cdots 1 \\ 0\,0 \cdots 0 \\ K_{k-2} \end{bmatrix}$.

For $m \geq k+2$,

$$\text{forb}(m, [2 \cdot K_k | H]) = \text{forb}(m, 2 \cdot K_k).$$

Extending $2 \cdot K_k$ by a column

Theorem Let $k \geq 2$ be a given integer and let $H = \begin{bmatrix} 1\,1 \cdots 1 \\ 0\,0 \cdots 0 \\ K_{k-2} \end{bmatrix}$.

For $m \geq k+2$,

$$\text{forb}(m, [2 \cdot K_k | H]) = \text{forb}(m, 2 \cdot K_k).$$

To verify the base case $m = k+2$, I tried for weeks to compute forb(m, H), but ultimately failed because I could not verify *that* base case of $m = k+1$. Eventually, we realized it sufficed to show forb$(k+1, H) \leq 2^{k+1} - k - 3$, and so the theorem was saved.

Extending $2 \cdot K_k$ by a column

While the previous theorem covers many examples of H for which $\text{forb}(m, [2 \cdot K_k | H]) = \text{forb}(m, 2 \cdot K_k)$, there can certainly be others. One other we have found:

Theorem For $m \geq 5$,

$$\text{forb}\left(m, \left[2 \cdot K_3 | \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right]\right) = \text{forb}(m, 2 \cdot K_3).$$

Extending $2 \cdot K_k$ by a column

While the previous theorem covers many examples of H for which $\text{forb}(m, [2 \cdot K_k | H]) = \text{forb}(m, 2 \cdot K_k)$, there can certainly be others. One other we have found:

Theorem For $m \geq 5$,

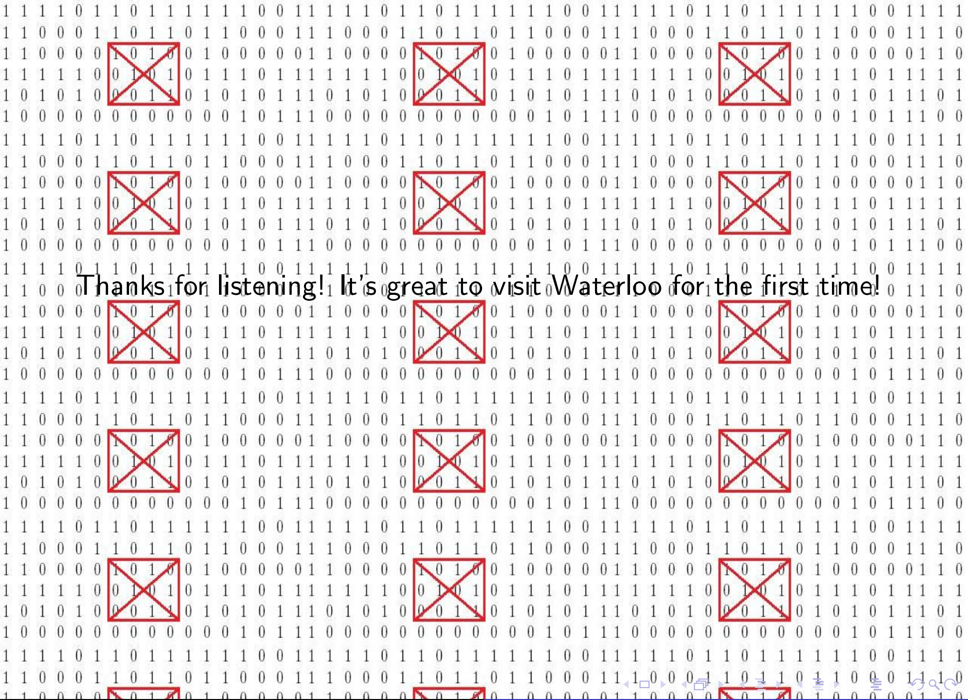
$$\text{forb}\left(m, \left[2 \cdot K_3 \middle| \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right]\right) = \text{forb}(m, 2 \cdot K_3).$$

This theorem uses a slightly different proof technique from the previous.

Open Questions for the Rest of the Summer

1. Is it true that $\text{forb}\left(m, \left[K_k \middle| \begin{bmatrix} 1\,1\cdots 1 \\ 1\,1\cdots 1 \\ 0\,0\cdots 0 \\ 0\,0\cdots 0 \\ K_{k-4} \end{bmatrix}\right]\right) = \text{forb}(m, K_k)$?

2. Is it true that there exists an m_0 such that for $m \geq m_0$,
$\text{forb}\left(m, \begin{bmatrix} 1\,1\cdots 1 \\ 0\,0\cdots 0 \\ K_{k-2} \end{bmatrix}\right) = \binom{m}{k-2} + \binom{m}{k-3} + \ldots + \binom{m}{0} + \binom{m}{m}$?



Thanks for listening! It's great to visit Waterloo for the first time!