On Methods of Induction in the study of Forbidden Configurations

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Abstract

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1 Introduction

The motivation to study forbidden configurations has its roots in graph theory. Mathematicians such as Erdős himself, asked what is the maximum number of edges a simple graph G on n vertices can have, before it no longer avoids a particular subgraph H as a subgraph. This question of graphs can be extended to the study of hypergraphs, where a *simple* hypergraph on m vertices is a family X of subsets of $\{1, 2, \ldots, m\}$. By convention X has no repeating elements and hence, in the language of hypergraphs, there are no "repeated" edges. The equivalent question in the theory of simple hypergraphs would be to determine the maximum number of hyperedges a simple hypergraph can have to ensure a certain subhypergraph is forbidden. Naturally, we can allow $X \subset 2^{[m]}$ to have repeated elements, thereby allowing for general hypergraphs with repeating edges; the analogous question would be to maximize the number of edges of a hypergraph while forbidding a family of subhypergraphs. Similarly, if we consider an m-rowed matrix with entries in $\{0, 1\}$, we would want to determine what the maximum number of columns it can have while avoiding a *configuration*, a term which will be defined shortly. The premise in studying forbidden configurations is generalizing the extremal subgraph problem to $(0, 1)$-matrices [1].

This task may seem quite daunting at first, especially when one lacks familiarity with techniques used in approaching the study of forbidden configurations. This survey gives a brief insight into proving results about forbidden configurations by using various methods of induction, which many mathematicians have employed. We will begin with a look at the *basic induction* in Section 2 that any student of mathematics has encountered. Next, we will consider the method known as *standard induction* in Section 3 which has been used ubiquitously throughout this subfield of extremal combinatorics. *Repeated induction*, found in Section 4, is an extension of standard induction. We will follow it up with a discussion about *multiplicity induction* in Section 5, which will consider matrices (and configurations) with repeated columns. Last, we will end with, for a lack of a better name, *sporadic induction*, in Section 6, a method of induction that has shown its usefulness but is not as central.

Below, we provide some preliminary definitions and remarks on the notation, which are most widely used in the study of forbidden configurations and will be used throughout this survey.

Definition 1.1 *Let $m \in \mathbb{N}$. We use the following notation:*

$$\begin{aligned} [m] &= \{1, 2, \dots, m\} \\ \binom{[m]}{k} &= \{S : S \subseteq [m], |S| = k\} \\ 2^{[m]} &= \{S : S \subseteq [m]\}. \end{aligned}$$

The latter is also known as the power set of $[m]$.

Definition 1.2 *We say that a $(0,1)$-matrix A is simple, if it has no repeated columns.*

Definition 1.3 *Given a $(0,1)$-matrix A, we denote the number of columns of A by $\|A\|$.*

Definition 1.4 *Let A be an $m \times n$ $(0,1)$-matrix. Let $S \subseteq [m]$ We denote by $A|_S$, the $|S| \times n$ submatrix of A of the rows $i \in S$.*

Now we can define our extremal problem.

Definition 1.5 *Given a $(0,1)$-matrix A, we say that a $(0,1)$-matrix F is a configuration of A, and denote it as $F \prec A$, if there exists a submatrix of A which is a row and column permutation of F.*

Definition 1.6 *Let $\mathcal{F}$ be a collection of $(0,1)$-matrices. Define*

$$\text{Avoid}(m, \mathcal{F}) = \{A \textit{ is } m - \textit{rowed} : F \nprec A, \forall F \in \mathcal{F}\}.$$

We call $\mathcal{F}$ a forbidden family of configurations.

Definition 1.7 *Let*

$$\textit{forb}(m, \mathcal{F}) = \max\{\|A\| : A \in \text{Avoid}(m, \mathcal{F})\}.$$

When possible, we seek exact values for forb(m, F); of course this may not always be the case and as such, one satisfies the query by seeking as accurate and sharp asymptotic bounds as possible.

Remark 1.8 *If F and G are two configurations such that $F \prec G$, then it follows that forb$(m, F) \leq$ forb(m, G).*

Proof: If $F \prec G$, then $A \in \text{Avoid}(m, F)$ implies that $A \in \text{Avoid}(m, G)$. So $\text{Avoid}(m, F) \subseteq \text{Avoid}(m, G)$, and hence, $\text{forb}(m, F) \leq \text{forb}(m, G)$. ∎

Let us first see some examples of forb(m, F) for some specific $(0,1)$-matrix F.

Example: Let's determine forb$(m, [1])$. Note that if $A \in \text{Avoid}(m, [1])$ with $\|A\| = \text{forb}(m, [1])$, then A cannot have 1 in any row or column. This leaves us with $A = \mathbf{0}_m$ since A has no repeating columns, and hence $\|A\| = 1$.

Example: Trying something a bit more challenging, suppose $A \in \text{Avoid}(m, \begin{bmatrix} 0 & 1 \end{bmatrix})$ with $\|A\| = \text{forb}(m, \begin{bmatrix} 0 & 1 \end{bmatrix})$. We determine what $\|A\|$ is. Note that on any row of A, we must avoid $\begin{bmatrix} 0 & 1 \end{bmatrix}$. Since A is simple, it cannot have any repeated columns. Suppose after some permutation of rows, the first column of A is $\mathbf{1}_i\mathbf{0}_{m-i}$ for some $0 \leq i \leq m$. Since this column cannot repeat, if A has another column, there exists a row $1 \leq j \leq m$ where we have either $\begin{bmatrix} 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \end{bmatrix}$, contradicting the assumption on A. So in fact $\|A\| = 1$ if $A \in \text{Avoid}(m, \begin{bmatrix} 0 & 1 \end{bmatrix})$.

For the next example, we introduce some notation:

Definition 1.9 *Denote by I_m the $m \times m$ identity matrix and denote by I_m^c the $(0,1)$-complement.*

Definition 1.10 *Denote by T_m the upper triangular $(0,1)$-matrix where the upper triangular entries are 1; that is, $T_{ij} = 1$ for $i \leq j$ and $T_{ij} = 0$ for $i > j$.*

Definition 1.11 *We denote with K_k, the $k \times 2^k$ simple $(0,1)$-matrix.*

Example: We attempt one more example, which will be a bit more non-trivial than the two above. Let $A \in \text{Avoid}(m, I_2)$ with $\|A\| = \text{forb}(m, I_2)$, where I_2 denotes the 2×2 identity matrix. We first note that $I_2 \prec K_2$, and so $\text{forb}(m, I_2) \leq \text{forb}(m, K_2)$. By Theorem 3.2 that we will prove in a later section, we conclude that

$$\|A\| \leq \text{forb}(m, K_2) = \binom{m}{1} + \binom{m}{0} = m + 1.$$

Next, we note that $[\mathbf{0}_m | T_m]$, the $m \times (m+1)$ upper triangular $(0,1)$-matrix, avoids I_2 as a configuration; therefore, $T_m \in \text{Avoid}(m, I_2)$. Thus, $\|A\| \geq \|[\mathbf{0}_m | T_m]\| \geq m + 1$. We conclude $\|A\| = m$.

Remark 1.12 *Thus, K_k is like the incidence matrix representation of the complete graph, with every possible $(0,1)$-string of length k, arranged as columns.*

Next, we introduce the matrix theoretic definition of what it means for a set $S \subseteq [m]$ to be shattered by a matrix A.

Definition 1.13 *Let A be a $(0,1)$-matrix. We say that $S \subseteq [m]$ is shattered by A, if $K_{|S|} \prec A|_S$. We say $\emptyset$ is shattered by A if and only if $\|A\| \geq 1$. Denote by $Sh(A)$, the sets $S \subseteq [m]$ shattered by A.*

The following product construction has proved useful.

Definition 1.14 *Let A be an $m_1 \times n_1$ simple matrix and B be an $m_2 \times n_2$ simple matrix. Denote by $A \times B$ the $(m_1 + m_2) \times (n_1 n_2)$ simple matrix where each column consists of a column of A placed on a column of B, done in every possible way.*

2 Basic Induction

In the theory of forbidden configurations and extremal set theory, the term "basic induction" will refer to the proof technique described as follows. Suppose A is a simple, m-rowed matrix; we identify any row and/or column permutation of A, with itself. The method of basic induction will require one to induct on the number of columns, n or rows, m. As with any method of induction, one checks that the base case, $n = 1$ in the case of inducting on the number of columns, holds and assumes the inductive hypothesis for all $1 \leq k < n$, and proving the claim for the case when $k = n$. To illustrate, suppose A is a simple m-rowed $(0,1)$-matrix with n columns; we claim that some statement $P(n)$ holds true for the matrix A, and would like to use the method of basic induction on the number of columns of A. We recognize that any row and column permutation of A is identified with A itself; as such, assuming that A has at least 2 columns, we have the following decomposition after permuting the rows and columns of A, so that row r has at least one 0 and one 1:

$$A = \begin{matrix} r \rightarrow \\ \\ \end{matrix} \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ & A_0 & & & A_1 & \end{bmatrix},$$

noting that $\|A_0\|, \|A_1\| \leq n - 1$. It is convenient to allow A_0 or A_1 be empty matrices, namely with 0 columns. In our first example, we will avoid this by assuming row r has at least one 0 and one 1. Here, we can invoke the inductive hypothesis on the submatrices A_0 and A_1, namely that the statement $P(\|A_0\|)$ and $P(\|A_1\|)$ holds true. We hope that invoking the inductive hypothesis on $\|A_0\|$ and $\|A_1\|$ is sufficient to conclude the claim for $\|A\|$. We note that we are not using any special properties of A_0 and A_1, with the exception that the two submatrices have fewer rows than A; this will be starkly contrasted when studying the method of standard induction, which will use the power of induction as well as certain special properties of submatrices in its decomposition to conclude claims about the matrix A.

This method of induction is the simplest method of induction and has been used in proving numerous important theorems in extremal set theory. One such result is the Shattered Set Lemma, which we shall state and prove in this section, giving both a matrix-theoretic and set-theoretic proof.

We first prove the Shattered Set Lemma for the case of $(0, 1)$-matrices.

Lemma 2.1 (Shattered Set Lemma, Pajor [8]) *Let A be a given m rowed matrix. Then $|Sh(A)| \geq ||A||$.*

Proof: First, let us consider the base case, when $||A|| = 1$. Trivially, the empty $(0, 1)$-matrix is shattered by A if $\|A\| \geq 1$ and so, $|Sh(A)| \geq ||A|| = 1$. Now assume inductive hypothesis for all $(0, 1)$-matrices A, with $||A|| = n$, with $1 \leq j < n$. Suppose that A is a $(0, 1)$-matrix, such that $||A|| = n$, with $n \geq 2$. . We permute the rows of A and find a row r so that row r contains at least one 0 and one 1. Such a row r exists, since by assumption $n \geq 2$. Decompose A in the following way:

$$A = \begin{matrix} r \rightarrow \\ \end{matrix} \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ & A_0 & & & A_1 & \end{bmatrix},$$

where $||A_0||, ||A_1|| \leq n-1$. As such, we can apply the inductive hypothesis and conclude that

$$|Sh(A_0)| \geq ||A_0||,$$
$$|Sh(A_1)| \geq ||A_1||.$$

We first remark that $||A|| = ||A_0|| + ||A_1||$, and by the inclusion–exclusion principle,

$$|Sh(A_0) \cup Sh(A_1)| + |Sh(A_0) \cap Sh(A_1)| = |Sh(A_0)| + |Sh(A_1)| \geq ||A_0|| + ||A_1||.$$

Next, we show that $|Sh(A)| \geq |Sh(A_0) \cup Sh(A_1)| + |Sh(A_0) \cap Sh(A_1)|$. Suppose $S \subseteq [m]$ with $S \in Sh(A_0) \cap Sh(A_1)$. By definition, we have that $K_{|S|} \prec A_0|_S$ and $K_{|S|} \prec A_1|_S$. Remarking that $r \notin S$, we observe that $K|_{|S \cup \{r\}|} \prec A|_{S \cup \{r\}}$, and so $S \cup \{r\} \in Sh(A)$. Also $S \cup \{r\} \notin Sh(A_0) \cup Sh(A_1)$. Hence, $Sh(A)$ contains at least $|Sh(A_0) \cap Sh(A_1)|$ more sets than $Sh(A_0) \cup Sh(A_1)$. Therefore, we conclude that $|Sh(A)| \geq |Sh(A_0) \cup Sh(A_1)| + |Sh(A_0) \cap Sh(A_1)|$, and hence

$$|Sh(A)| \geq ||A_0|| + ||A_1|| \geq ||A||.$$

This completes the proof of the matrix version of Shattered Set Lemma. ∎

Now we give a proof of the Shattered Set Lemma for the case of sets.
Here, we state the requisite definitions and the statement of the aforementioned lemma, and use the method of basic induction to prove it.

Definition 2.2 *Let $\mathcal{U} \subseteq 2^{[m]}$. We say that $\mathcal{U}$ shatters a set $S \subseteq [m]$ if*

$$\{T \cap S : T \in \mathcal{U}\} = 2^S$$

In other words, we hope to recover all subsets of S by intersecting S with all subsets $T \in \mathcal{U}$.

Definition 2.3 *For $\mathcal{U} \subseteq 2^{[m]}$, let $Sh(\mathcal{U})$ be the collection of all T that are shattered by $\mathcal{U}$.*

Now we state and prove the Shattered Set Lemma for sets.

Lemma 2.4 (Shattered Set Lemma, Pajor [8]) *Let $\mathcal{U} \subseteq 2^{[m]}$, with $\mathcal{U} \neq \emptyset$. Then $|Sh(\mathcal{U})| \geq |\mathcal{U}|$.*

Proof: We prove this theorem, as aforementioned, by using the method of basic induction. We know that $\emptyset \in Sh(\mathcal{U})$ for $\mathcal{U} \neq \emptyset$. So $|Sh(\mathcal{U})| \geq |\mathcal{U}|$ holds for the base case. Now we assume the inductive hypothesis for all $\mathcal{U} \subseteq 2^{[m]}$ such that $|\mathcal{U}| \in \{1, 2, \ldots, n\}$ and prove the lemma for when $|\mathcal{U}| = n+1$. Let $t \in [m]$ such that t is contained in at least one, but not all $S \in \mathcal{U}$. We let $\mathcal{U}_0 = \{S \in \mathcal{U} : t \notin S\}$ and $\mathcal{U}_1 = \{S \in \mathcal{U} : t \in S\}$. Then it is clear that $\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1$, where $\sqcup$ denotes a disjoint union. Now, we observe that $|\mathcal{U}_0|, |\mathcal{U}_1| < n+1$ and hence, by the inductive hypothesis $|Sh(\mathcal{U}_0)| \geq \mathcal{U}_0, |Sh(\mathcal{U}_1)| \geq \mathcal{U}_1$. Applying a case of the inclusion-exclusion principle, we obtain that

$$\begin{aligned} |Sh(\mathcal{U}_0) \cup Sh(\mathcal{U}_1)| + |Sh(\mathcal{U}_0) \cap Sh(\mathcal{U}_1)| &= |Sh(\mathcal{U}_0)| + |Sh(\mathcal{U}_1)| \\ &\geq |\mathcal{U}_0| + |\mathcal{U}_1| \\ &= |\mathcal{U}| \end{aligned}$$

We want to show that

$$|Sh(\mathcal{U}_0) \cup Sh(\mathcal{U}_1)| + |Sh(\mathcal{U}_0) \cap Sh(\mathcal{U}_1)| \leq |Sh(\mathcal{U})|. \quad (1)$$

We note that since $Sh(\mathcal{U}_0) \cap Sh(\mathcal{U}_1) \subseteq Sh(\mathcal{U}_0) \cup Sh(\mathcal{U}_1)$, if $S \in Sh(\mathcal{U}_0) \cup Sh(\mathcal{U}_1) - Sh(\mathcal{U}_0) \cap Sh(\mathcal{U}_1)$, the it contributes exactly one unit to the left in (1) and one unit to the right in (1). If $S \in Sh(\mathcal{U}_0) \cap Sh(\mathcal{U}_1)$, it contributes exactly 1 unit to the left in (1) but 2 units to the right in (1), namely S and $S \cup \{t\}$. Since $Sh(\mathcal{U}_0) \cap Sh(\mathcal{U}_1), Sh(\mathcal{U}_0) \cup Sh(\mathcal{U}_1) \subseteq Sh(\mathcal{U})$ we obtain the inequality in (1). ∎

If $S \in Sh(\mathcal{U}_0) \cap Sh(\mathcal{U}_1)$, we note that $t \notin S$; if it were the case that $t \in S$, then we would obtain that $\{T \cap S : T \in \mathcal{U}_0\} \subsetneq S$ since $t \notin T$ for any $T \in \mathcal{U}_0$. Therefore, it follows that $S \cup \{t\} \notin Sh(\mathcal{U}_0)$. Now, S, $S \cup \{t\} \in Sh(\mathcal{U})$. So we have contributions of 2 units to the right side of the inequality in (1) while there is only a contribution of 1 unit to the left side of the inequality in (1). This proves our claim.

Definition 2.5 *For notational simplicity, we let*

$$G_{6\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$F_7 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Theorem 2.6 *[6] Suppose F is a 6-rowed $(0,1)$-matrix. Then $F \prec G_{6\times 3}$ if and only if forb$(m, F) = \Theta(m^2)$. Otherwise, if $F \not\prec G_{6\times 3}$, then forb$(m, F)$ is $\Omega(m^3)$.*

Theorem 2.7 *[6] For F_7 defined above, forb$(m, F_7) = \Theta(m^2)$. For any $(0,1)$-column α, forb$(m, [F_7|\alpha])$ is $\Omega(m^3)$.*

We will discuss aspects of the proof that forb(m, F_7) is $O(m^2)$ later in the paper. The following is a lemma which proves one direction of Theorem 2.6.

Lemma 2.8 *[6] Suppose F is a 6-rowed $(0,1)$-matrix such that $F \not\prec G_{6\times 3}$. Then forb$(m, F)$ is $\Omega(m^3)$.*

Proof: First, we remark that without loss of generality, we can assume that the columns of F have a column sum of exactly 3. Why so? Well first, suppose that F has a column α with column sum 2 or less. Then we note that $\alpha \not\prec I^c \times I^c \times I^c$, since the $3-$fold product has columns with column sum strictly bigger than 2. Likewise, if α is a column of F with column sum greater than 4; then $\alpha \not\prec I \times I \times I$, and the $3-$fold product has columns of column sum at most 3. In both of these cases, we have a construction with $\Omega(m^3)$ columns and hence, forb$(m, F) = \Omega(m^3)$.

So let us assume that F is a 6-rowed matrix so that it has columns only of column sum 3. We consider the following two cases first:

$$F_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

We claim that $F_1 \not\prec I \times I \times I$. This follows from the fact that the 3 – fold product $I \times I \times I$, is symmetric in I and hence, I has to cover $\begin{bmatrix} 1 & 1 \end{bmatrix}$, contradicting the fact that I has no row with a row sum of 2 or more. Next, we claim that $F_2 \not\prec I \times I \times T$. We argue this by first noting that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \not\prec T$. So T can cover one or more of the first three rows, or one or more of the bottom three rows, but no combination of rows from the top three or bottom three. Suppose without loss generality that T covers $i \in \{1, 2, 3\}$ of the first three rows. Then we note that $I \times I$ has to cover $6 - i$ of the rows and in particular, I has to cover at least 2 of the bottom 3 rows. But notice that $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \not\prec I$ since I has no column with column sum 2 or more. So indeed we have a construction, $I \times I \times T$ on $\Omega(m^3)$ rows which does not contain the configuration F_2.

Anstee and Sali showed that what remains is to check the following six cases. First, observe that

$$F = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \not\prec I^c \times I^c \times I^c.$$

Notice that the 3 – fold product is symmetric in I^c. Moreover, observe that no pair of the last 4 rows of F can be covered by a single I^c, since I^c has no pair of rows with row sum 1 or less. Hence, no triple or 4 – tuple of the last four rows of F can be covered by a single I^c. As such, I^c must cover one of the first two rows, combined with one of the last four rows. Since there are only two rows of F with column sum more than 2, and four rows of F with column sum 1, we run into a problem, as some pair of the last four rows must be covered by I^c, contradicting our argument above.

Next, we argue that if

$$F = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then $F \not\prec I \times I \times I$. The case for this claim is in the same flavour as that of the above matrix. We notice that this new F has the first four rows with row sum equal to 4, while the last two rows are of row sum equal to zero. Since I does not have any pair of rows with row sum equal to 4, no pair, triple or 4 – tuple of the first four rows of F can be covered by I. As such, any row from the top 4 rows, must be combined with one of the bottom 2 rows to be covered by I; since there are 2 remaining rows of row sum 4, we have that these remaining 2 rows must be covered by I, which contradicts the argument we made above.

To shorten the arguments as for the above two cases, we list here the remainder of the cases where F is a 6-rowed, 3-columned simple matrix which has columns of sum exactly 3:

$$\begin{bmatrix}1&1&1\\1&1&0\\1&0&0\\0&1&0\\0&0&1\\0&0&1\end{bmatrix}\not\prec I^c\times I^c\times I^c,\qquad\begin{bmatrix}1&1&0\\1&1&0\\1&0&1\\0&1&1\\0&0&1\\0&0&0\end{bmatrix}\not\prec I\times I\times I,\qquad\begin{bmatrix}1&1&0\\1&0&1\\1&0&0\\0&1&1\\0&1&0\\0&0&1\end{bmatrix}\not\prec I\times I\times T$$

and not to forget,

$$\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}=G_{6\times3}.$$

This shows that if F is a 6-rowed matrix, 3-columned matrix that avoids $G_{6\times3}$, then $\mathrm{forb}(m,F)=\Omega(m^3)$. As such, any F with 4 or more columns which avoids $G_{6\times3}$ is such that $\mathrm{forb}(m,F)=\Omega(m^3)$. Anstee and Sali showed that the only 6-rowed, 4-columned matrix which contains, on every triple of columns, $G_{6\times3}$ as a configuration, is the following:

$$F=\begin{bmatrix}1&1&1&1\\1&1&0&0\\1&0&1&0\\0&1&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix}\not\prec T\times T\times T.$$

Since F is 4-columned, $F\not\prec G_{6\times3}$ and Anstee and Sali have a construction, namely the 3-fold product $T\times T\times T$, that avoids F.

To sum up, we have shown that if F is a 6-rowed matrix such that $F\not\prec G_{6\times3}$, then $\mathrm{forb}(m,F)=\Omega(m^3)$. ■

Lemma 2.9 below will be useful in proving the bound on $G_{6\times3}$, from the bound on F_7.

Lemma 2.9 *[6] Suppose F is a $(0,1)$-simple matrix of the following form:*

$$F = \begin{bmatrix} 0 & \dots & 0 \\ 1 & \dots & 1 \\ & F' & \end{bmatrix}.$$

Next, suppose that F_0 and F_1 are defined as follows:

$$F_0 = \begin{bmatrix} 0 & \dots & 0 \\ & F' & \end{bmatrix}, \quad F_1 = \begin{bmatrix} 1 & \dots & 1 \\ & F' & \end{bmatrix}.$$

Then, we claim that

$$\mathit{forb}(m, F) \le \mathit{forb}(m-1, F_0) + \mathit{forb}(m-1, F_1).$$

Proof: Suppose that A has 3 or more rows. Let $A \in \text{Avoid}(m, F)$ and without loss of generality, we assume that $\|A\| = \text{forb}(m, F)$. Find a row r, permute the rows and columns of A so that we have a row r with at least one 0 and one 1, at the top, as in the following decomposition:

$$A = \begin{matrix} r \rightarrow \end{matrix} \begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ & A' & & & A'' & \end{bmatrix}.$$

Noting that $\|A\| = \|A'\| + \|A''\|$, we observe the following: $F_1 \nprec A'$, since otherwise:

$$F \prec \begin{bmatrix} 0 & \dots & 0 \\ & F & \end{bmatrix} \prec \begin{bmatrix} 0 & \dots & 0 \\ & A' & \end{bmatrix} \prec A,$$

contradicting the hypothesis that $A \in \text{Avoid}(m, F)$. Therefore, we must have that $A' \in \text{Avoid}(m-1, F)$. Using a similar argument, we can show that $A'' \in \text{Avoid}(m-1, F_0)$. By the above, we conclude that

$$\text{forb}(m, F) = \|A\| = \|A_0\| + \|A_1\| \le \text{forb}(m-1, F_0) + \text{forb}(m-1, F_1). \tag{2}$$

■

Corollary 2.10 *[6] We have that $\mathit{forb}(m, G_{6\times 3})$ is $O(m^2)$.*

Proof: Permute the rows of $G_{6\times 3}$ so that the row of 0's and 1's become the first two rows, and define G_0 and G_1 as follows:

$$G_{6\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ & G' & \end{bmatrix}, \quad G_0 = \begin{bmatrix} 0 & 0 & 0 \\ & G' & \end{bmatrix}, \quad G_1 = \begin{bmatrix} 1 & 1 & 1 \\ & G' & \end{bmatrix}.$$

Now, from Lemma 2.9, we know that

$$\text{forb}(m, G_{6\times 3}) \le \text{forb}(m-1, G_0) + \text{forb}(m-1, G_1) \le 2 \cdot \text{forb}(m-1, F_7),$$

where the last inequality follows from the fact that $G_0 \prec F_7$ and $G_1 \prec F_7$. We use Theorem 2.6 to conclude $\text{forb}(m-1, F_7)$ to be $O(m^2)$ and thereby completing the proof.

■

3 Standard Induction

Here, we give a description of what is known as "standard induction", a proof technique frequently used in showing numerous results about forbidden configurations. Let $A \in \text{Avoid}(m, \mathcal{F})$. We recall that any row and column permutation of A will be considered as the matrix A itself. Suppose we permute the columns of A such that we have the following:

$$A = \left[\begin{array}{c} 0\ldots0 \quad 1\ldots1 \\ \hline A' \end{array}\right].$$

We note that the matrix A' need not be simple; for example, if

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

then removing the first row yields $A' = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, which is not simple. That being said, we observe that if A is a simple matrix, then any column of A' repeats at most twice. As such, we can permute rows and columns of A, an m-rowed matrix, and find a row r of A so that we have the following:

$$A = \begin{matrix} r \rightarrow \\ \end{matrix} \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & & C_r & C_r & & D_r \end{bmatrix},$$

where C_r consists of all the columns of A' which repeat. We will call $[B_r C_r D_r]$ and C_r the *inductive children* of A, and refer to A as the *parent*.

By construction, $[B_r C_r D_r]$ and C_r are simple $m-1$-rowed matrices, where the former is $[B_r C_r D_r] \in \text{Avoid}(m-1, \mathcal{F})$. Note that $\|A\| = \|[B_r C_r D_r]\| + \|C_r\|$. Additionally, we note that we could find an r so that $\|C_r\|$ is minimal, which is relevant in some proof not discussed here. [1]

We now consider what the $(m-1)$-rowed simple matrix C_r must avoid.

Remark 3.1 *[3] Suppose that $\mathcal{F}$ is some family of $(0,1)$-matrices and $A \in \text{Avoid}(m, \mathcal{F})$. Let $F \in \mathcal{F}$ and find a row s so that after some row and column permutation of F, we have the following:*

$$F = \begin{matrix} s \rightarrow \\ \end{matrix} \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B(F)_s & & C(F)_s & C(F)_s & & D(F)_s \end{bmatrix}.$$

We remark that if $[B(F)_s C(F)_s D(F)_s] \prec C_r$, then $F \prec A$, contradicting the hypothesis that $A \in \text{Avoid}(m, \mathcal{F})$. Let

$$\mathcal{F}' = \{[B(F)_s C(F)_s D(F)_s] : F \in \mathcal{F}, s \in row(F)]\}$$

where $row(F)$ is the number of rows of F. Then if $A \in \text{Avoid}(m, \mathcal{F})$, it follows that $C_r \in \text{Avoid}(m-1, \mathcal{F}')$. [1].

In fact, we can often reduce $\mathcal{F}'$ somewhat, as we will see in an example later.

We now prove a version of the Sauer bound, using the standard induction method, as described above.

Theorem 3.2 (Sauer [9], Perles-Shelah [10], Vapnik-Chervonenkis [11]) *Let k be given. Then we have that*

$$forb(m, K_k) = \binom{m}{k-1} + \ldots + \binom{m}{0}$$

Proof: We first prove that forb$(m, K_k) \leq \binom{m}{k-1} + \ldots + \binom{m}{0}$ by inducting on m, the number of rows. Let A be a simple m-rowed matrix, with $A \in Avoid(m, K_k)$, and without loss of generality, suppose that $||A|| = forb(m, K_k)$. Using the above method described in remark 3.1 permute the columns and rows of the matrix A so that we have

$$A = \begin{matrix} r \rightarrow \\ \end{matrix} \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & & C_r & C_r & & D_r \end{bmatrix}.$$

Remark that $||A|| = ||[B_r C_r D_r|| + ||C_r||$ Next, we observe that $K_{k-1} \not\prec C_r$; otherwise, if indeed $K_{k-1} \prec C_r$, then we have that

$$K_k = \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ & K_k & & & K_k & \end{bmatrix} \prec \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ & C_r & & & C_r & \end{bmatrix} \prec A$$

contradicting our original hypothesis that $A \in Avoid(m, K_k)$. Hence $C_r \in Avoid(m-1, K_{k-1})$ by remark 3.1. By inductive hypothesis, since C_r is an $(m-1)$-rowed simple matrix,

$$||C_r|| \leq \binom{m-1}{k-2} + \ldots + \binom{m-1}{0}$$

and $[B_r C_r D_r] \in forb(m-1, K_k)$, which gives

$$||[B_r C_r D_r]|| \leq \binom{m-1}{k-1} + \ldots + \binom{m-1}{0}.$$

Recall Pascal's identity for binomial coefficients:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

for all $1 \leq k \leq n-1$. Combining the above remark with the bounds on $||C_r||$ and

$||[B_rC_rD_r]||$, we have that

$$\begin{aligned}||A|| &\leq \binom{m-1}{k-1} + \ldots + \binom{m-1}{1} + \binom{m-1}{0} \\ &+ \binom{m-1}{k-2} + \ldots + \binom{m-1}{0} \\ &= \left(\binom{m-1}{k-1} + \binom{m-1}{k-2}\right) + \ldots + \left(\binom{m-1}{1} + \binom{m-1}{0}\right) + 1 \\ &= \binom{m}{k-1} + \ldots + \binom{m}{0}.\end{aligned}$$

So we have shown an upper bound on $\text{forb}(m, K_k)$. Now we show that $\text{forb}(m, K_k) \geq \binom{m}{k-1}+\ldots+\binom{m}{0}$, by constructing a matrix simple $(0,1)$-matrix A which does not contain K_k as a subconfiguration. We observe that K_k contains the column $\mathbf{1_k}$. So, to construct a matrix which does not contain K_k as a subconfiguration, we ensure that the constructed matrix avoids $\mathbf{1_k}$. We observe that for $0 \leq j \leq k$, there are exactly $\binom{m}{j}$ distinct columns of length k with column sum j, since there are exactly $\binom{m}{j}$ ways of choosing where to place j 1's and $m-j$ 0's. Let K_m^j denote the $m \times \binom{m}{j}$ sized simple matrix, consisting of all possible (0,1)-columns of column sum j. Now, we let

$$A := [K_m^0|K_m^1|\cdots|K_m^{k-2}|K_m^{k-1}].$$

We note that A has exactly $\binom{m}{k-1} + \ldots + \binom{m}{0}$ columns and since there does not exist a column in A with column sum k. Note that $\mathbf{1_k} \not\prec A$. Hence, we conclude that $K_k \not\prec A$ and therefore, $A \in \text{Avoid}(m, K_k)$. This yields the fact that

$$\text{forb}(m, K_k) \geq \|A\| = \binom{m}{k-1} + \ldots + \binom{m}{0},$$

thereby proving the other direction of the inequality. ■

First, we make the following remark:

Remark 3.3 *[1] If F and G are configurations so that $F \prec G$, then $forb(m, F) \leq forb(m, G)$.*

Proof: If $F \prec G$, then having $A \in Avoid(m, F)$ gives that $A \in Avoid(m, G)$, and we obtain that $Avoid(m, F) \subseteq Avoid(m, G)$. Hence, it follows that

$$forb(m, F) = max\{||A|| : A \in Avoid(m, F)\} \leq max\{||A|| : A \in Avoid(m, G)\} = forb(m, G)$$

yielding the desired inequality. ■

Now we state the following definition and theorems and show how to use the argument of standard induction to prove Theorem 2.6.

Definition 3.4 *Denote by F_7 the following matrix:*

$$F_7 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{3}$$

First, we fix a notation.

Definition 3.5 *For $i \in \{1, 2, 3, 4, 5\}$, let H_i denote the simple matrix attained by deleting row i of F_7 and removing any repeated columns.*

By Lemma 3.1, if $A \in \text{Avoid}(m, F_7)$ then $C_r \in \text{Avoid}(m, \{H_1, H_2, H_3, H_4, H_5\}$.
First, let's look at H_1. Deleting row 1 of F_7, yields the following matrix:

$$H_1' = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Observe that H_1' has no repeating columns, and hence C_1 has no columns. So in fact $[B_1 C_1 D_1] = H_1' = H_1$.

Deleting row 2 in F_7 yields:

$$H_2' = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We note that the column $\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$, appears twice and hence, $C_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$. So we conclude that H_2 is the 4-rowed matrix with only one copy of the repeating column from H_2', namely C_2:

$$H_2 = [B_2 C_2 D_2] = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Next, we determine what H_3 is. Removing the 3^{rd} row from F_7 yields H_3' which has two columns which repeat twice:

$$H_3' = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We notice that the columns $\begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}^T$ and $\begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix}^T$ repeat twice and therefore, by our standard decomposition method described above,

$$C_3 = \begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\\ 0 & 0\end{bmatrix}.$$

H_3 is therefore the matrix attained by removing the third row of F_7 and C_3, namely:

$$H_3 = \begin{bmatrix}1 & 1 & 0 & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Note that H_3^c is the same configuration as H_3. Determining H_4 and H_5 requires the same procedure, and hence have just been stated below:

$$H_4 = \begin{bmatrix}0 & 1 & 1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix},\ H_5 = \begin{bmatrix}1 & 1 & 0 & 1 & 0\\ 1 & 0 & 1 & 1 & 1\\ 0 & 1 & 0 & 1 & 1\\ 0 & 0 & 1 & 0 & 1\end{bmatrix} [6].$$

We make a further few remarks which will aid in proving $\text{forb}(m, F_7)$ is $O(m)$.

Remark 3.6 *For any configuration F, forb$(m, F) =$ forb(m, F^c).*

Remark 3.7 *We note that $H_3^c = H_3$, $H_4 = H_1^c$ and $H_5 = H_2^c$.[6]*

In fact, we observe that if we permute the second row with the first row in H_1 to obtain

$$H_1 = \begin{bmatrix}0 & 1 & 1 & 0 & 0 & 1\\ 1 & 0 & 1 & 1 & 1 & 1\\ 0 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}.$$

Then we notice columns 2, 3, 4 and 5 give us precisely H_3. So in fact $H_3 \prec H_1$. Hence, if $A \in \text{Avoid}(m, H_3)$, then $A \in \text{Avoid}(m, H_1)$. Next, we observe that columns 2, 3, 4, and 6 of H_4 form H_3. Therefore, $H_3 \prec H_4$ and so, $\text{Avoid}(m, H_3) \subset \text{Avoid}(m, H_4)$. From these remarks we can conclude the following:

$$\text{forb}(m, \{H_1, H_2, H_3, H_4, H_5\}) \le \text{forb}(m, \{H_2, H_3, H_5\}).$$

This is the reduction of $\mathcal{F}$ that we referred to in Remark 3.1.

Theorem 3.8 *[6] forb*$(m, \{H_2, H_3, H_5\}) \leq 7m$.

The proof is rather involved and tricky. We shall still use this result to show

$$\text{forb}(m, F_7) \leq 7m^2.$$

Proof: [Theorem 2.6]
Suppose that $A \in \text{Avoid}(m, F_7)$ with $\|A\| = \text{forb}(m, F_7)$, and find a row r to perform standard induction, and find $[B_r C_r D_r$ and C_r, which are $(m-1)$-simple matrices [6]:

$$\begin{aligned}\|A\| &= \|[B_r C_r D_r]\| + \|C_r\| \\ &\leq \text{forb}(m-1, F_7) + \text{forb}(m-1, \{H_1, H_2, H_3, H_4, H_5\}) \\ &\leq 7(m-1)^2 + \text{forb}(m-1, \{H_2, H_3, H_5\}) \\ &\leq 7(m-1)^2 + 7(m-1) \\ &\leq 7m^2. \quad \blacksquare\end{aligned}$$

4 Repeated Induction

Repeated induction is a special case of standard induction for forbidden configuration. We began in section 3 with a simple m-rowed matrix, performed the standard decomposition and observed certain properties of the inductive children. In the case of repeated induction however, the forbidden configurations of the inductive children retain some property related to the original family of forbidden configurations, or in other words, the parent.

For example, consider the case where $A \in \text{Avoid}(m, K_k)$, and assume that $\|A\| = \text{forb}(m, K_k)$. After performing the standard decomposition on A, we obtain

$$A = \begin{matrix} r_1 \rightarrow \\ \\ \end{matrix} \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_{r_1} & & C_{r_1} & C_{r_1} & & D_{r_1} \end{bmatrix},$$

where $[B_{r_1} C_{r_1} D_{r_1}]$ is a simple $(m-1)$-rowed matrix which avoids K_k; that is, $[B_{r_1} C_{r_1} D_{r_1}] \in \text{Avoid}(m-1, K_k)$, while C_{r_1} is a simple $(m-1)$-owed matrix which avoids K_{k-1}: ie. $C_{r_1} \in \text{Avoid}(m-1, K_{k-1})$. The justification for the latter is that if $K_{k-1} \prec C_{r_1}$, then it follows that

$$K_k = \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ & K_{k-1} & & & K_{k-1} & \end{bmatrix} \prec \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ & C_{r_1} & & & C_{r_1} & \end{bmatrix} \prec A,$$

contradicting our initial hypothesis that $A \in \text{Avoid}(m, K_k)$. Thus $K_{k-1} \not\prec C_{r_1}$. We observe that C_{r_1}, an inductive child of A, retains a very similar property to that of A, namely that if $A \in \text{Avoid}(m, K_k)$, then $C_{r_1} \in \text{Avoid}(m-1, K_{k-1})$. Inductively, if we find a row r_2 of C_{r_1} and perform standard decomposition, we obtain

$$C_{r_1} = \begin{matrix} r_2 \rightarrow \end{matrix} \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B'_{r_2} & & C'_{r_2} & C'_{r_2} & & D'_{r_2} \end{bmatrix},$$

where $C'_{r_2} \in \text{Avoid}(m-2, K_{k-2})$ [5]. This procedure can be repeated.

To summarize, we observe that A, avoided K_k as a configuration and after performing the standard decomposition, the inductive child C_{r_1} necessarily avoided K_{k-1}. Consequently, performing standard decomposition on C_{r_1}, we find C'_{r_2} where $C'_{r_2} \in \text{Avoid}(m-2, K_{k-2})$; notice that K_{k-2} is the complete $(0,1)$-matrix on $k-2$ rows. Repeating this procedure, for $1 \leq t \leq k$, we can find $C_{r_t}^{(t-1)}$ so that $C_{r_t}^{(t-1)} \in \text{Avoid}(m-t, K_{k-t})$. Repeated induction is characterized by invoking this fact that the inductive children have properties very similar to forbidden configurations of the parent.

We now provide examples of where such an induction proves to be of terrific use, where the inductive children preserve some sort of property of the parent. Let us begin by proving useful lemmas which use repeated induction.

Lemma 4.1 (Anstee and Meehan [5]) *For $m \geq 3$,*

$$\text{forb}(m, \mathcal{F}_2) = \text{forb}(m, K_2) = m + 1$$

where $\mathcal{F}_2 = \{[K_2|\mathbf{1}_2], [K_2|\mathbf{0}_2], [K_2|\mathbf{1}_1\mathbf{0}_1]\}$.

Proof: We first make remark that for $m = 2$, our claim does not hold. By the Sauer bound,

$$\text{forb}(2, K_2) = \binom{2}{1} + \binom{2}{0} = 2 + 1 = 3.$$

We note that K_2 is a simple, 2-rowed matrix which avoids every $F \in \mathcal{F}_2$ and hence, $\text{forb}(2, \mathcal{F}_2) \geq 4$. Therefore, for $m = 2$, we have a strict inequality:

$$\text{forb}(2, \mathcal{F}_2) > \text{forb}(2, K_2).$$

Next we make the following observation: since for all $F \in \mathcal{F}_2, K_2 \prec F$, we have that if $A \in \text{Avoid}(m, K_2)$, then $A \in \text{Avoid}(m, \mathcal{F}_2)$, and hence $\text{forb}(m, K_2) \leq \text{forb}(m, \mathcal{F}_2)$. Now, assume for contradiction, that we have strict inequality; that is, $\text{forb}(m, K_2) < \text{forb}(m, \mathcal{F}_2)$. Let $G \in \text{Avoid}(m, \mathcal{F}_2)$ and without loss of generality, assume that $\|G\| = \text{forb}(m, \mathcal{F}_2)$. By our assumption, $\|G\| > m+1$ and since $m \geq 3$, we have that $\|G\| \geq 5$. Now, we observe that $K_2 \prec G$; otherwise, we have that G is a simple m-rowed matrix that avoids the configuration K_2 and hence $\|G\| = \text{forb}(m, K_2)$, a contradiction to our initial assumption. So indeed $K_2 \prec G$ and on some pair of rows $\{i, j\}$, we have that $K_2 \prec G|_{\{i,j\}}$. Moreover, we observe that $\|G|_{\{i,j\}}\| = \|G\|$ and so we have a column of $G|_{\{i,j\}}$, in addition to the four columns of K_2. This column can only be one of $\mathbf{1}_2, \mathbf{0}_2$ or $\mathbf{1}_1\mathbf{0}_1$ and so $G \notin \text{Avoid}(m, \mathcal{F}_2)$, a contradiction to our initial hypothesis. So we conclude that indeed $\text{forb}(m, K_k) = \text{forb}(m, \mathcal{F}_2)$ when $m \geq 3$. ■

Lemma 4.2 *[5] For* $m \geq 4$,

$$\text{forb}(m, \mathcal{F}_3) = \text{forb}(m, K_3)$$

where $\mathcal{F}_3 = \{[K_3|\mathbf{1}_2\mathbf{0}_1], [K_3|\mathbf{1}_1\mathbf{0}_2]\}$.

Proof: By the Sauer bound, we can conclude that $\text{forb}(3, K_3) = 7$. But we note that K_3 is a 3-rowed matrix which avoids every $F \in \mathcal{F}_3$, and hence $\text{forb}(3, \mathcal{F}_3) \geq 8$. So we deduce that our lemma does not hold for $m = 3$.

In general, we note that $\text{forb}(m, \mathcal{F}_3) \geq \text{forb}(m, K_3)$. We use induction on m to prove that $\text{forb}(m, \mathcal{F}_3) \leq \text{forb}(m, K_3)$. First consider the base case when $m = 4$. Note that since $K_3 \prec F$ for every $F \in \mathcal{F}_3$, it follows that $\text{forb}(4, K_3) \leq \text{forb}(4, \mathcal{F}_3)$. Now for contradiction, assume strict inequality: $\text{forb}(4, K_3) < \text{forb}(4, \mathcal{F}_3)$. Let $G \in \text{Avoid}(4, \mathcal{F}_3)$ and $\|G\| = \text{forb}(4, \mathcal{F}_3)$. As before, we must have that $K_3 \prec G$ and G has at least $\text{forb}(4, K_3) + 1 = 12$ columns; therefore, on some triple of rows G contains the configuration K_3 and since there are at least 12 columns, we have at least 4 columns in this triple of rows in addition to the 8 columns of K_3. Now we remark that if G is to avoid all $F \in \mathcal{F}_3$, then only $\mathbf{1}_3$ and $\mathbf{0}_3$ can be placed in the remaining columns of the triple of rows which contain K_3. Since there are at least 4 remaining columns, at least one of $\mathbf{1}_3$ or $\mathbf{0}_3$ must appear at least twice in the 4 remaining columns; but observe that both $\mathbf{1}_3$ and $\mathbf{0}_3$ appear once in K_3 and hence, at least one of $\mathbf{1}_3$ or $\mathbf{0}_3$ will appear at least three times in the triple of rows containing K_3. But this will be impossible if G is simple. Given this contradiction, we conclude $\text{forb}(4, K_3) = \text{forb}(4, \mathcal{F}_3)$.

Now we assume the inductive hypothesis; that is for all $4 \leq k \leq m-1$, we have that

$$\text{forb}(k, K_3) = \text{forb}(k, \mathcal{F}_3).$$

Let $A \in \text{Avoid}(m, \mathcal{F}_3)$ with $\|A\| = \text{forb}(m, \mathcal{F}_3)$. Then $K_3 \prec A$. We find a row r and perform the standard decomposition to obtain $[B_rC_rD_r]$ and C_r, where the former is an $(m-1)$-rowed simple matrix which avoids $\mathcal{F}_3$ and the latter is a simple $(m-1)$-rowed matrix avoiding $\mathcal{F}_2$, since $\mathcal{F}_2$ is the family of inductive children of $\mathcal{F}_3$. This argument follows from our explanation of the repeated induction above, where we saw that the inductive children retain some special property of the parent. Now by inductive hypothesis, we have that

$$\|[B_rC_rD_r]\| \leq \text{forb}(m-1, \mathcal{F}_3) = \text{forb}(m-1, K_3).$$

From the above lemma, we have that

$$\|C_r\| \leq \text{forb}(m-1, \mathcal{F}_2) = \text{forb}(m-1, K_2).$$

Putting these inequalities together, we obtain the desired bound:

$$\text{forb}(m, \mathcal{F}_3) = \|A\| = \|[B_rC_rD_r]\| + \|C_r\| \leq \text{forb}(m-1, K_3) + \text{forb}(m-1, K_2) = \text{forb}(m, K_3).$$

We already have that $\text{forb}(m, \mathcal{F}_3) \geq \text{forb}(m, K_3)$ and so we have the conclusion. ■

Theorem 4.3 *[5] Let $k \geq 4$, and $2 \leq p \leq k-2$. Then, for $m \geq k+1$, we have that*

$$\mathit{forb}(m, [K_k|\mathbf{1}_p\mathbf{0}_{k-p}]) = \mathit{forb}(m, K_k).$$

Proof: We induct on both k and m. First, consider the base case when $k = 4$ and $m = 5$. Then $p = 2$ and so we have to show that

$$\text{forb}(5, [K_4|\mathbf{1}_2\mathbf{0}_2]) = \text{forb}(5, K_4).$$

We note that since $K_4 \prec [K_4|\mathbf{1}_2\mathbf{0}_2]), \text{forb}(5, K_4) \leq \text{forb}(5, [K_4|\mathbf{1}_2\mathbf{0}_2])$. Assume strict inequality for the sake of contradiction; that is:

$$\text{forb}(5, K_4) < \text{forb}(5, [K_4|\mathbf{1}_2\mathbf{0}_2]).$$

Let $A \in \text{Avoid}(5, [K_4|\mathbf{1}_2\mathbf{0}_2])$ and $\|A\| = \text{forb}(5, [K_4|\mathbf{1}_2\mathbf{0}_2])$. Then $\|A\| > 26$ and hence, on a quadruple of rows S_4 of A, we have 16 columns, containingK_4 as a configuration and we still have at least 11 columns on this quadruple of rows which need to be decided. It is clear that no permutation of $\mathbf{1}_2\mathbf{0}_2$ can be placed in any of these remaining rows, since any row and column permutation would yield $[K_4|\mathbf{1}_2\mathbf{0}_2]$. So the only $(0,1)$-columns of length four which can be placed in the remaining columns of the quadruple of rows are either $\mathbf{1}_4$ and $\mathbf{0}_4$, as well as any permutation of $\mathbf{1}_3\mathbf{0}_1$ and $\mathbf{1}_1\mathbf{0}_3$. We observe that there are 8 ways of choosing a permutation of $\mathbf{1}_3\mathbf{0}_1$ and $\mathbf{1}_1\mathbf{0}_3$ which gives a total of 10 possible distinct columns on rows of S_4. We note that since A is simple and already has K_4, we cannot have any of these columns appear three times or more on the quadruple of rows. But this yields a contradiction since we have at least 11 columns to populate with distinct $(0,1)$-columns, and there are only 10 possible choices. So we have that

$$\text{forb}(5, K_4) = \text{forb}(5, [K_4|\mathbf{1}_2\mathbf{0}_2]).$$

Next, assume that for all $5 \leq n \leq m-1$ and $k = 4$, we have that $\text{forb}(n, K_4) = \text{forb}(n, [K_4|\mathbf{1}_2\mathbf{0}_2])$. Let $\|A\| = \text{forb}(m, [K_4|\mathbf{1}_2\mathbf{0}_2])$. Next, we find a row r and perform the standard decomposition on A and obtain $(m-1)$-rowed $[B_rC_rD_r]$ which avoids $[K_4|\mathbf{1}_2\mathbf{0}_2]$, and $(m-1)$-rowed C_r, which avoids $[K_3|\mathbf{1}_2\mathbf{0}_1]$ and $[K_3|\mathbf{1}_1\mathbf{0}_2]$. We provide a justification for this claim as follows. If $[K_3|\mathbf{1}_2\mathbf{0}_1] \prec C_r$, then we have that

$$[K_4|\mathbf{1}_2\mathbf{0}_2] \prec \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ & [K_3|\mathbf{1}_2\mathbf{0}_1] & & & [K_3|\mathbf{1}_2\mathbf{0}_1] & \end{bmatrix} \prec \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ & C_r & & & C_r & \end{bmatrix} \prec A,$$

contradicting our initial hypothesis. A similar argument holds for $[K_3|\mathbf{1}_1\mathbf{0}_2]$ and thus, $C_r \in \text{Avoid}(m, \{[K_3|\mathbf{1}_2\mathbf{0}_1], [K_3|\mathbf{1}_1\mathbf{0}_2]\})$. We notice how the inductive child C_r of A preserved a property very similar to that of A after the standard decomposition was employed. Now since $C_r \in \text{Avoid}(m, \{[K_3|\mathbf{1}_2\mathbf{0}_1], [K_3|\mathbf{1}_1\mathbf{0}_2]\})$, we have by the above lemma that

$$\|C_r\| \leq \text{forb}(m-1, \{[K_3|\mathbf{1}_2\mathbf{0}_1], [K_3|\mathbf{1}_1\mathbf{0}_2]\}) = \binom{m-1}{2} + \binom{m-1}{1} + \binom{m-1}{0}.$$

Moreover, by inductive hypothesis, it follows that

$$\begin{aligned}\|[B_rC_rD_r]\| &\le \text{forb}(m-1,[K_4|\mathbf{1}_2\mathbf{0}_2]) = \text{forb}(m-1,K_4)\\ &= \binom{m-1}{3}+\binom{m-1}{2}+\binom{m-1}{1}+\binom{m-1}{0}.\end{aligned}$$

Last, we conclude that

$$\begin{aligned}\|A\| &= \|[B_rC_rD_r]\| + \|C_r\|\\ &\le \binom{m-1}{3}+\binom{m-1}{2}+\binom{m-1}{1}+\binom{m-1}{0}+\\ &\binom{m-1}{2}+\binom{m-1}{1}+\binom{m-1}{0}\\ &= \binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}\\ &= \text{forb}(m,K_4)\end{aligned}$$

where the second last equality follows by Pascal's identity.

Next, we let $m = k+1$ and prove that $\text{forb}(k+1,[K_k|\mathbf{1}_p\mathbf{0}_{k-p}]) = \text{forb}(k+1,K_k)$. First, we remark that

$$\text{forb}(k+1,K_k) = \binom{k+1}{k-1}+\ldots+\binom{k+1}{0} = 2^{k+1}-(k+1)-1,$$

where the last equality comes from the combinatorial fact that $2^{k+1} = \sum_{i=0}^{k+1}\binom{k+1}{i}$. Now we assume for contradiction that $\text{forb}(k+1,[K_k|\mathbf{1}_p\mathbf{0}_{k-p}]) > \text{forb}(k+1,K_k)$. Then, on some $k-$ tuple of rows of the $k+1$ rows, we have that there is a configuration K_k. This means that 2^k of the columns are determined but it still leaves us

$$2^{k+1}-(k+1)-1-2^k = 2^k-(k+1)$$

columns to determine. Now we want to avoid $[K_k|\mathbf{1}_p\mathbf{0}_{k-p}]$ and hence none of the remaining columns can be any permutation of $\mathbf{1}_p\mathbf{0}_{k-p}$. There are exactly $\binom{k}{p}$ distinct permutations of $\mathbf{1}_p\mathbf{0}_{k-p}$ and hence, we have $2^k-\binom{k}{p}$ distinct columns to place on the k tuple of rows containing K_k. But we observe that since $k > 2$, then $k+1 < \binom{k}{p}$ and therefore, $2^k-\binom{k}{p} < 2^k-(k+1)$; so in fact we must have a column appearing three times in the k tuple of rows containing K_k, violating the simplicity assumption.

Last, we take $m > k+1 > 5$ and $k > 4$, and show that $\text{forb}(m,[K_k|\mathbf{1}_p\mathbf{0}_{k-p}]) = \text{forb}(m,K_k)$. Without loss of generality, we may assume the inequality $\text{forb}(m,[K_k|\mathbf{1}_p\mathbf{0}_{k-p}]) \ge \text{forb}(m,K_k)$ is clear since $K_k \prec [K_k|\mathbf{1}_p\mathbf{0}_{k-p}]$. Assume next that

$A \in \text{Avoid}(m, [K_k|\mathbf{1}_p\mathbf{0}_{k-p}])$ with $\|A\| = \text{forb}(m, [K_k|\mathbf{1}_p\mathbf{0}_{k-p}])$, and take for contradiction, that $\text{forb}(m, K_k) < \text{forb}(m, [K_k|\mathbf{1}_p\mathbf{0}_{k-p}])$. Find a row r and perform standard decomposition on the matrix A to obtain $(m-1)$-rowed simple matrices $[B_rC_rD_r]$ and C_r. Since $[B_rC_rD_r] \in \text{Avoid}(m-1, [K_k|\mathbf{1}_p\mathbf{0}_{k-p}])$ by the inductive hypothesis, we conclude that

$$\|[B_rC_rD_r]\| \leq \binom{m-1}{k-1} + \ldots + \binom{m-1}{0}.$$

Also $C_r \in \text{Avoid}(m-1, [K_{k-1}|\mathbf{1}_{p-1}\mathbf{0}k-p]$ and so by induction on k we have that

$$\|C_r\| \leq \binom{m-1}{k-2} + \cdots \binom{m-1}{0}.$$

Now $\|A\| = \|[B_rC_rD_r]\| + \|C_r\|$ and so by Theorem 3.2, we have that $\|A\| \leq \text{forb}(m, K_k)$. ■

Above, we observed the power of repeated induction, and invoking the inductive hypothesis when the inductive children preserve some property similar to that of the parent. We provide another example in the study of forbidden configuration, where repeated induction makes an appearance. We begin with some notation.

Definition 4.4 *We fix the following notation for the next example:*

$$E_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The theorem that we would like to prove, with the help of repeated induction is the following:

Theorem 4.5 (Anstee and Fleming [2]) *Let F be a k-rowed matrix which has the following property:*

1. *there exist rows i_1 and j_1 so that $E_1 \not\prec F|_{\{i_1,j_1\}}$.*
2. *there exist rows i_2 and j_2 so that $E_2 \not\prec F|_{\{i_2,j_2\}}$.*
3. *there exist rows i_3 and j_3 so that $E_3 \not\prec F|_{\{i_3,j_3\}}$.*

Then, it follows that forb$(m, F) = O(m^{k-2})$. If, on the contrary, there exists a $k \in \{1, 2, 3\}$ for which $E_k \prec F|_{\{i,j\}}$ for every pair of rows i and j, then we have that forb$(m, F) = \Theta(m^{k-1})$.

In proving the above theorem, we require the result of the following lemma, which will use repeated induction; that is, we will observe that in invoking the inductive hypothesis, we will make use of the fact that the inductive children have a specific property related to that of the parent.

Lemma 4.6 *[2] Let $k \geq 2$ be given and let F_1, F_2, and F_3 be k-rowed simple matrices, which are not necessarily distinct, such that*

1. *there exist rows i_1 and j_1 so that all the columns of $F_1|_{\{i_1,j_1\}}$ are configurations of* $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$
2. *there exist rows i_2 and j_2 so that all the columns of $F_2|_{\{i_2,j_2\}}$ are configurations of* $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
3. *there exist rows i_3 and j_3 so that all the columns of $F_3|_{\{i_3,j_3\}}$ are configurations of* $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Noting that the F_i need not to be distinct, if A is a simple m-rowed matrix with the property that $F_i \nprec A$ for any $i \in \{1,2,3\}$, then it follows that

$$\|A\| \leq 2\left[\binom{m}{k-2} + \ldots + \binom{m}{0}\right].$$

Proof: We prove the lemma by inducting on m and k. For the first base case, let $k = 2$. By assumption, each F_i is 2-rowed and moreover, since $E_i \nprec F_i$ for any $i \in \{1,2,3\}$. If we want to avoid $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in F_1 and F_1 is 2-rowed, the columns of F_1 can be $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; moreover, since F_1 is simple, the maximal 2-rowed simple matrix avoiding E_1 is

$$F_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Using a symmetric argument, we can, without loss of any generality, assume that

$$F_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Last, to avoid the 2×2 identity matrix in F_3, we can have the columns of F_3 consisting of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and exactly one of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, but not both; hence, the maximal 2-rowed simple matrix avoiding $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is

$$F_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, we consider the base case $m = 2$, when $k = 2$. We want to show that if $A \in \text{Avoid}(m, \{F_1, F_2, F_3\})$ and $\|A\| = \text{forb}(m, \{F_1, F_2, F_3\})$, then $\|A\| \leq 2$. Since A is a simple 2-rowed $(0,1)$-matrix and it avoids F_1, F_2 and F_3, we can have that the distinct columns of A can be at most two of $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$, $\begin{bmatrix}0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$, and so we have our conclusion that $\|A\| \le 2 = 2\binom{2}{0}$. Assume that, when $k=2$, for all $2 \le n \le m-1$, we have that

$$\text{forb}(n, \{F_1, F_2, F_3\}) \le 2\left[\binom{n}{0}\right] = 2$$

Now assume that $A \in \text{Avoid}(m, \{F_1, F_2, F_3\})$ with $\|A\| = \text{forb}(m, \{F_1, F_2, F_3\})$. We first eliminate a few simple cases. If A has a row of 0's, after row permutations, we have the following decomposition:

$$A = \begin{bmatrix} 0 & \dots & 0 \\ & A' & \end{bmatrix}$$

where A' is an $(m-1)$-rowed simple matrix and hence by our inductive hypothesis, we conclude that

$$\|A\| = \|A'\| \le 2\left[\binom{m-1}{0}\right] = 2\left[\binom{m}{0}\right] = 2.$$

By a symmetrical argument, if we have a row of 1's, we have that $\|A\| \le 2$. Now assume that there exist rows i and j of A so that one row is the $(0,1)$-complement of the other. Permute the rows and columns of A to obtain the following decomposition:

$$\left[\begin{array}{cc} 0\dots0 & 1\dots1 \\ 1\dots1 & 0\dots0 \\ \hline \multicolumn{2}{c}{A'} \end{array}\right].$$

Noting that $\|A\| = \|A'\|$, we remark that

$$\left[\begin{array}{cc} 1\dots1 & 0\dots0 \\ \hline \multicolumn{2}{c}{A'} \end{array}\right]$$

is $(m-1)$-rowed and simple. We justify why indeed it is simple; suppose we have that a column in the above $(m-1)$-rowed matrix repeats. Then it must be of the form $\begin{bmatrix}1 & 1\\ \alpha & \alpha\end{bmatrix}$ or $\begin{bmatrix}0 & 0\\ \alpha & \alpha\end{bmatrix}$ and indeed we have that either $\begin{bmatrix}0 & 0\\ 1 & 1\\ \alpha & \alpha\end{bmatrix}$ or $\begin{bmatrix}1 & 1\\ 0 & 0\\ \alpha & \alpha\end{bmatrix}$ are columns of A, violating the simplicity of A. Therefore,

$$\left[\begin{array}{cc} 1\dots1 & 0\dots0 \\ \hline \multicolumn{2}{c}{A'} \end{array}\right]$$

is indeed simple. Now, applying our inductive hypothesis, we obtain that

$$\|A\| = \|A'\| \le 2\binom{m-1}{0} = 2.$$

Assume now that the rows of A have none of the above three properties, namely no rows of 0's, no rows of 1's, and no two complementary rows. For any pair of rows i and j, we observe that to avoid F_i for $i \in \{1,2,3\}$ in A, we can have at most two of $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$, $\begin{bmatrix}0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$ occur in $A|_{\{i,j\}}$. But for any choice of two columns, one of the three eliminated cases arise. For example, if we have that the columns of $A|_{\{i,j\}}$ consist of $\begin{bmatrix}0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$, then we have either a row of 0's or row of 1's to delete. If we have that the columns of $A|_{\{}i,j\}$ consist of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$, then we have that the rows are $(0,1)$-complements of each other and hence, we can delete one of the two rows to obtain one of the eliminated cases. This contradicts our hypothesis that A did not have the property observe in the eliminated cases. So, we conclude that $\|A\| \le 2\binom{m}{0} = 2$. We remark for the reader that we have yet to employ repeated induction, which shall be used in the case of a general $k > 2$.

Now we assume that $k > 2$, where F_1, F_2 and F_3 are k-rowed simple matrices, avoiding E_1, E_2 and E_3 respectively. If $m < k$, then we remark that A will have fewer rows than F_1, F_2 and F_3 and hence can never have those as a configuration. So we assume $k \le m$. Let $A \in \text{Avoid}(m, \{F_1, F_2, F_3\})$, with $\|A\| = \text{forb}(m, \{F_1, F_2, F_3\})$. Find a row r, permute the rows and columns of A and perform the standard decomposition as described before:

$$A = \begin{matrix} r \rightarrow \\ \\ \end{matrix} \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & & C_r & C_r & & D_r \end{bmatrix},$$

where $[B_r C_r D_r]$ and C_r are simple, $(m-1)$-rowed matrices. Note that $[B_r C_r D_r]$ avoids $\{F_1, F_2, F_3\}$, and therefore,

$$\|[B_r C_r D_r]\| \le 2\left[\binom{m-1}{k-2} + \ldots + \binom{m-1}{0}\right].$$

Now, for each $n \in \{1,2,3\}$, we select a row from $[k] \setminus \{i_n, j_n\}$, denoting it t_n, and letting $F_n' = F_n|_{[k]\setminus t_n}$. Then we observe that $E_n \not\prec F_n'$ for all $n \in \{1,2,3\}$; indeed, we see that the inductive child C_r of A has a property similar to that of A, which is that it avoids structures which contain E_1, E_2 and E_3 as configurations. So C_r is an $(m-1)$-rowed simple matrix, avoiding the $(k-1)$-rowed $\{F_i'\}_{i=1}^3$. By our inductive hypothesis, we have that

$$\|C_r\| \le 2\left[\binom{m-1}{k-3} + \ldots + \binom{m-1}{0}\right].$$

Noting that $\|A\| = \|[B_r C_r D_r]\| + \|C_r\|$ and using the above results, along with Pascal's identity, we obtain the desired:

$$\|A\| \le 2\left[\binom{m-1}{k-2} + \ldots + \binom{m-1}{0}\right]. \quad \blacksquare$$

Now we prove the main theorem.

Proof: [**Theorem 4.5, Anstee and Fleming**]. Let F be such that:

1. there exist rows i_1 and j_1 so that $E_1 \not\prec F|_{\{i_1,j_1\}}$
2. there exist rows i_2 and j_2 so that $E_2 \not\prec F|_{\{i_2,j_2\}}$
3. there exist rows i_3 and j_3 so that $E_3 \not\prec F|_{\{i_3,j_3\}}$,

Here we note that F has the properties described in the hypothesis of Lemma 4.6 that we have just proven, where $F = F_1 = F_2 = F_3$. Hence, if $A \in \text{Avoid}(m, F)$ such that $\|A\| = \text{forb}(m, F)$, then $\|A\| = O(m^{k-1})$. What remains to be shown is that $\|A\| = \Omega(m^{k-1})$.

Now assume that there exists $n \in \{1, 2, 3\}$ for which $E_n \prec F|_{\{}i, j\}$ for all $\{i, j\}$. We need to find a construction with $\Omega(m^{k-1})$ columns avoiding F. First, we note the following. E_1 is not contained in the identity complement, I^c, since no column of I^c has two rows contained 0's. Likewise, $E_2 \not\prec I$ since no column of I has two rows with 1's. Last, $E_3 \not\prec T$, where T is any upper triangular $(0, 1)$-matrix; if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \prec T$ where $a = d = 1$ and $c = 0$, then necessarily we have that $b = 1$. Now suppose that for some $n \in \{1, 2, 3\}$, we have that $E_n \prec F|_{\{i,j\}}$, for every pair of rows i and j, and let $G^n \in \{I, I^c, T\}$ such that $E_n \not\prec G^n$. In fact, without loss of generality, we can take $G^1_\ell = I^c_\ell, G^2_\ell = I_\ell$ and $G^3_\ell = T_\ell$. We want to construct a matrix that is m-rowed and $c \cdot m^{k-1}$-columned, for some $c > 0$, not containing F as a configuration.

Recall that $\lfloor \frac{m}{k-1} \rfloor$ is the number of copies of $k - 1$ in m, and we let $r = m-(k-1)\cdot\lfloor \frac{m}{k-1} \rfloor$. The notation G^n_ℓ represents the $\ell\times\ell$ identity, identity complement or upper triangular matrix, depending on what G^n_ℓ is. Now let

$$H = \underbrace{G^n_{\lfloor \frac{m}{k-1} \rfloor} \times \cdots \times G^n_{\lfloor \frac{m}{k-1} \rfloor}}_{(k-2)} \times G^n_r$$

which is a $(k - 1)$-fold product with m rows. We note that if $F \prec H$, then on some pair of rows i and j of H, we have to have that $E_n \prec H|_{\{i,j\}}$, leading to a contradiction. This is because H is constructed as $(k - 1)$-fold product of G^n, and hence at least 2 of the k rows containing F must come from one of the product terms, which does not contain E_n as a configuration. So, we have constructed the matrix H with m rows and $\Omega(m^{k-1})$ columns which does not contain F as a configuration. Now we can conclude that if F has the properties:

1. there exist rows i_1 and j_1 so that $E_1 \not\prec F|_{\{i_1,j_1\}}$
2. there exist rows i_2 and j_2 so that $E_2 \not\prec F|_{\{i_2,j_2\}}$

3. there exist rows i_3 and j_3 so that $E_3 \not\prec F|_{\{i_3,j_3\}}$,

then $\text{forb}(m, F) = \Theta(m^{k-1})$. ■

Next, we generalize the above theorem, but first we state a few definitions and a result proven by Balogh and Bollobás.

Definition 4.7 *For $k \geq 2$, define*

$$\begin{aligned} E_1(k) &= [\mathbf{1}_k|I_k^c], \\ E_2(k) &= [\mathbf{0}_k|I_k], \\ E_3(k) &= [\mathbf{0}_k|T_k] \end{aligned}$$

Theorem 4.8 (Balogh-Bollobás [7]) *For $k \geq 2$, forb$(m, \{E_1(k), E_2(k), E_3(k)\}) = c_k$, for some constant $c_k > 0$; that is forb$(m, \{E_1(k), E_2(k), E_3(k)\})$ is O(1).*

With the aid of the Balogh-Bollobás result above, we prove a generalization of Theorem 4.5.

Theorem 4.9 *[2] Let $k \geq 2$ and $p \geq k$ be given. Suppose that F_1, F_2 and F_3 are simple p-rowed matrices with the following property:*

1. *there exist k rows $S_1 \subseteq [p]$ so that all the columns of $F_1|S_1$ are contained in $E_1(k)$*
2. *there exist k rows $S_2 \subseteq [p]$ so that all the columns of $F_2|S_2$ are contained in $E_2(k)$*
3. *there exist k rows $S_3 \subseteq [p]$ so that all the columns of $F_3|S_3$ are contained in $E_3(k)$*

Then forb$(m, \{F_1, F_2, F_3\}) = O(m^{p-k})$.

Proof: We prove this by inducting on p and m, and use the Balogh-Bollobás result from above. Suppose first for our base case, that $p = k$. Without loss of generality, we may take $F_1 = E_1(k)$, $F_2 = E_2(k)$, and $F_3 = E_3(k)$. Why can we do that? Fix an $i \in \{1, 2, 3\}$. Next, we observe that since $p = k$, any k sized subset S_i of $[k]$ is indeed $[k]$ and if we assume the hypothesis, then all the columns of $F_i|_{S_i} = F_i$ are contained as columns in $E_i(k)$. Since the maximal k-rowed simple F_i with distinct columns contained in $E_i(k)$ is indeed $E_i(k)$, we assume that $F_i(k) = E_i(k)$. Then, by the Balogh-Bollobás result above, we conclude that

$$\text{forb}(m, \{F_1, F_2, F_3\}) = O(m^{k-k}) = O(1).$$

Now we assume inductive hypothesis for $p > k$ and $m > n \geq p$: if F_1, F_2 and F_3 are simple p-rowed matrices so that they satisfy the hypothesis in the theorem, then

$$\text{forb}(n, \{F_1, F_2, F_3\}) \leq c_k(n^{p-k})$$

for some $c_k > 0$. Assume that $A \in \text{Avoid}(m, \{F_1, F_2, F_3\})$ so that $\|A\| = \text{forb}(m, \{F_1, F_2, F_3\})$. Find a row r and perform the standard decomposition on A and obtain the following:

$$A = \begin{matrix} r \rightarrow \end{matrix} \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r^A & & C_r^A & C_r^A & & D_r^A \end{bmatrix},$$

where $[B_r^A C_r^A D_r^A]$ is a simple $(m-1)$-rowed matrix avoiding F_1, F_2 and F_3. What requires more work is to determine what the $(m-1)$-rowed simple matrix C_r^A avoids, since it avoids more than F_1, F_2 and F_3. Fix an $i \in \{1,2,3\}$ and fix a row $t_i \in [p] \setminus S_i$. We perform the standard decomposition on F_i choosing row t_i as the row to remove and obtain:

$$F_i = \begin{matrix} t_i \rightarrow \end{matrix} \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_{t_i}^{F_i} & & C_{t_i}^{F_i} & C_{t_i}^{F_i} & & D_{t_i}^{F_i} \end{bmatrix}.$$

Let $F'_{t_i} = [B_{t_i}^{F_i} C_{t_i}^{F_i} D_{t_i}^{F_i}]$. Then by standard induction, we have that $C_r^A \in \text{Avoid}(m-1, F'_i)$ for all $i \in \{1,2,3\}$. Now we appeal to the inductive hypothesis in the case of $[B_r^A C_r^A D_r^A]$ and C_r^A. First, note that $\|A\| = \|[B_r^A C_r^A D_r^A]\| + \|C_r^A\|$. Second, we remark that by the Balogh-Bollobás result, $\text{forb}(m, \{E_1(p), E_2(p), F_3(p)\}) \leq c_p$, for some constant $c_p > 0$. Next, note that since F_1, F_2 and F_3 are simple p-rowed matrices, there are 2^p distinct columns and hence, $\text{forb}(p, \{F_1, F_2, F_3\}) \leq 2^p$. We let $C_p = \max\{c_p, 2^p\}$. Then, by the inductive hypothesis on $(m-1)$, we obtain that

$$\|[B_r^A C_r^A D_r^A]\| \leq \text{forb}(m-1, \{F_1, F_2, F_3\}) \leq C_p(m-1)^{p-k}.$$

Last, we use inductive hypothesis on the $(m-1)$-rowed simple matrix C_r^A and the $(p-1)$-rowed simple matrices F'_1, F'_2 and F'_3. We notice that for each $i \in \{1,2,3\}$, t_i was chosen from rows not contained in S_i and therefore, F'_i are simple $(p-1)$-rowed matrices which satisfy the hypotheses in the theorem; that is, $F'_i|_{S_i}$ has all the columns contained as columns in $E_i(k)$. Invoking our inductive hypothesis, we can find a constant C'_{p-1} so that

$$\|C_r^A\| \leq \text{forb}(m-1, \{F'_1, F'_2, F'_3\}) \leq C'_{p-1}(m-1)^{p-1-k}.$$

Putting it with our above remark, we conclude that:

$$\begin{aligned} \|A\| &= \|[B_r^A C_r^A D_r^A]\| + \|C_r^A\| \\ &\leq \text{forb}(m-1, \{F_1, F_2, F_3\}) + \text{forb}(m-1, \{F'_1, F'_2, F'_3\}) \\ &\leq C'_p(m-1)^{p-k} + C'_{p-1}(m-1)^{p-1-k} \\ &\leq C''_p m^{p-k} \end{aligned}$$

for some constant $C''_p > 0$. We remark that the constant may be extremely large with respect to p, but is independent of m. So we have shown that

$$\text{forb}(m, \{F_1, F_2, F_3\}) = O(m^{p-k}). \qquad \blacksquare$$

Observe that in the above proof, the inductive child C_r^A of A avoided structures very similar to the structures avoided by A; as a result, we were able to invoke the power of repeated induction and conclude our theorem. We state a corollary of the above theorem, where the proof is essentially the same.

Corollary 4.10 *[2] Let $k \geq 2$ and $p \geq k$ be given. Suppose that F is a simple, p-rowed matrix with the following property: there exist k-sized subsets S_1, S_2 and S_3 so that every $i \in \{1,2,3\}$, every column of $F|_{S_i}$ is contained as a column of $E_i(k)$. Then*

$$forb(m, F) = O(m^{p-k}).$$

Proof: The proof of the theorem comes as a direct consequence of the above theorem. In particular, we note that in the previous theorem, there were no assumptions made about the distinctness of the simple p-rowed matrices, F_1, F_2, and F_3 and therefore, by assuming that $F = F_1 = F_2 = F_3$, we obtain the result quite easily. ∎

An additional part of the theorem that we have not mentioned is the following fact: If the k – sized sets S_1, S_2, and S_3 have the following property:

1. $S_1 \cap S_2 = \emptyset$
2. $S_1 \cap S_3 \subseteq \min S_3$
3. $S_2 \cap S_3 \subseteq \max S_3$

where $\max S_3$ is the largest element of S_3, then F has the property of being a boundary case. In particular, if α is a p-rowed column such that α is not a column in F, then

$$\text{forb}(m, [F|\alpha]) = \Omega(m^{p-k+1}).$$

We will not provide the proof of this part of the theorem, as it involves constructions and does not provide insight into the use of repeated induction. The proof can be found in [2].

5 Multiplicity Induction

The previous three sections have been describing inductive methods to use when one is attempting to prove results about forbidding configurations in simple $(0,1)$-matrices. The natural progression is to investigate how the asymptotics change when forbidding configurations in $(0,1)$-matrices which are not simple. We begin the discussion of what we call multiplicity induction, with a few definitions and remarks.

Definition 5.1 *Let α be a column of a $(0,1)$-matrix, A. We denote by $\mu(\alpha, A)$, the number of times α appears as a column in A. We call $\mu(\alpha, A)$, the multiplicity of α in A.*

Remark 5.2 *We note that a matrix A is simple if $\mu(\alpha, A) \leq 1$ for all $(0,1)$-columns α.*

Definition 5.3 *If A is a $k \times \ell$ $(0,1)$-matrix that is not necessarily simple, we define supp(A) to be the k-rowed simple matrix consisting of all columns α of A, such that*

$$\mu(\alpha, supp(A)) = 1 \Leftrightarrow \mu(\alpha, A) \geq 1.$$

Remark 5.4 *Observe that if A is a $(0,1)$-matrix so that for every column α of A, $\mu(\alpha, A) \leq t$, then $A \prec t \cdot supp(A)$ and so $\|A\| \leq t \cdot \|supp(A)\|$.*

Definition 5.5 *Let A be a $(0,1)$-matrix. We say that A is t-simple if for every column α of A, we have that $\mu(\alpha, A) \leq t$.*

Definition 5.6 *We denote with* $\text{Avoid}(m, F, t-1)$ *the following:*

$$\text{Avoid}(m, F, t-1) = \{A : A \textit{ is } m\textit{-rowed } (t-1) \textit{ simple }, F \not\prec A\};$$

as before, we define

$$forb(m, F, t-1) = \max\{\|A\| : A \in \text{Avoid}(m, F, t-1)\}.$$

Remark 5.7 *[4] For any family of configurations $\mathcal{F}$,*

$$forb(m, \mathcal{F}) \leq forb(m, \mathcal{F}, s) \leq s \cdot forb(m, \mathcal{F}).$$

We make some important remarks which will be very useful in some theorems that we want to prove to show the usefulness of multiplicity induction.

Lemma 5.8 *[4] Let $A \in \text{Avoid}(m, F, t-1)$, with $\|A\| = forb(m, F, t-1)$. Find a row r, permute the columns and rows of A and obtain the following decomposition:*

$$A = \begin{matrix} r \rightarrow \\ \end{matrix} \begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ & G & & & H & \end{bmatrix}.$$

Proof: By assumption, we note that A is $(t-1)$ – simple. Then $\mu(\alpha, G) \leq t-1$ and $\mu(\alpha, H) \leq t-1$. We prove this lemma by contradiction. Assume there is some column α of G with multiplicity greater than or equal to t. Hence, it follows that

$$\underbrace{\begin{bmatrix} 0 & \dots & 0 \\ \alpha & \dots & \alpha \end{bmatrix}}_{\geq t} \prec \begin{bmatrix} 0 & \dots & 0 \\ & G & \end{bmatrix} \prec A,$$

showing that $\mu(\begin{bmatrix} 0 \\ \alpha \end{bmatrix}, A) \geq t$, thereby contradicting the initial supposition that A is $(t-1)$ – simple. Hence, for α columns of G, we have $\mu(\alpha, G) \leq t-1$. The exact same argument is used to show that for all columns α of H, $\mu(\alpha, H) \leq t-1$.

5.1 Standard Decomposition for Multiplicity Induction

Recall that in standard induction and repeated induction, we took a simple, m-rowed $(0,1)$-matrix A, which avoided some family of configurations $\mathcal{F}$, performed standard decomposition on it, and determined what the inductive children on $(m-1)$-rows, $[BCD]$ and C, avoided. This allowed us to use simplicity of the inductive children as well as the inductive hypothesis, to conclude results about the assumed matrix A.

Here, we give a similar method of decomposition a $(t-1)$-simple matrix on m rows, so that we can use the inductive hypothesis on the inductive children to conclude results about the initial matrix.

Let $A \in \text{Avoid}(m, F, t-1)$ with $\|A\| = \text{forb}(m, F, t-1)$. Find a row r and obtain the following decomposition:

$$A = \begin{matrix} r \rightarrow \\ \end{matrix} \begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ & G & & & H & \end{bmatrix}. \tag{4}$$

Define

$$R = \{\alpha : \mu(\alpha, [GH]) \geq t\}$$

which is the collection of all columns α of $[GH]$ which have a multiplicity greater than or equal to t in $[GH]$. We form C_r, an $(m-1)$-rowed matrix, that is $(t-1)$-simple, in the following way. Let C_r be formed by taking $\alpha \in R$ and so that

$$\forall \alpha \in R,\ \mu(\alpha, C_r) = \min\{\mu(\alpha, G), \mu(\alpha, H)\}.$$

Then we obtain the following decomposition:

$$A = \begin{matrix} r \rightarrow \\ \end{matrix} \begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ B_r & & C_r & C_r & & D_r \end{bmatrix}, \tag{5}$$

where $[B_r C_r D_r]$ and C_r are $(t-1)$-simple $(m-1)$-rowed matrices. The following is a justification as to why the former is $(t-1)$-simple, as it is not very clear. We note that for every α column of $[B_r C_r D_r]$,

$$\begin{aligned} \mu(\alpha, [B_r C_r D_r]) &= \mu(\alpha, [GH]) - \mu(\alpha, C_r) \\ &= \mu(\alpha, G) + \mu(\alpha, H) - \mu(\alpha, C_r) \\ &= \mu(\alpha, G) + \mu(\alpha, H) - \min\{\mu(\alpha, G), \mu(\alpha, H)\} \\ &\leq \max\{\mu(\alpha, G), \mu(\alpha, H)\} \\ &\leq (t-1) \end{aligned}$$

where the last inequality comes from Lemma 5.8.

Also,

$$\mu(\alpha, C_r) = \min\{\mu(\alpha, G), \mu(\alpha, H)\} \leq \max\{\mu(\alpha, G), \mu(\alpha, H)\} \leq (t-1).$$

Next, we give an example to illustrate this method of standard decomposition. Take A to be the following:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We note that the above A is 3-simple, since for every column α of A, we have that $\mu(\alpha, A) \leq 3$. Also A is not 2-simple since the last column has multiplicity 3. Then using (4) and $r = 1$, we obtain

$$G = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we remark that $\mu(\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T, [GH]) = 4 > 3$. Observing that for no other α column of G or H, $\mu(\alpha, [GH]) \geq 4$, we conclude that $R = \{\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T\}$. From the above description of forming the inductive child C, we have that

$$\mu(\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T, C) = \min\{\mu(\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T, G), \mu(\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T, H)\} = \min\{1, 3\} = 1.$$

Then we have the following decomposition of A:

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

with

$$A = \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_1 & & C_1 & C_1 & & D_1 \end{bmatrix}.$$

Last, we notice that $[B_1 C_1 D_1]$ is $3-$simple, as is C_1.

We will use this standard decomposition for $(t-1)$-simple matrices in lemmas and theorems that follow, which will require the use of multiplicity induction. But first, we need to state some facts that arise from this method of decomposing a matrix. [4]

Suppose that $A \in \text{Avoid}(m, \mathcal{F}, t-1)$, with $\|A\| = \text{forb}(m, \mathcal{F}, t-1)$. Perform the standard decomposition for $(t-1)$-simple matrices, as described above, and remark that

$$\|A\| = \|[B_r C_r D_r]\| + \|C_r\|.$$

As in the section of standard decomposition, we have that $[B_r C_r D_r] \in \text{Avoid}(m-1, \mathcal{F}, t-1)$. What remains to be determined is what C_r avoids. We first begin by claiming that for any $F \in \mathcal{F}$, $\text{supp}(F) \not\prec C$. Notice that for every column α of C_r, by construction, we have it so that $\mu(\alpha, [GH]) \geq t$. If $\text{supp}(F) \prec C_r$, every column of $\text{supp}(F)$ has multiplicity at least t in $[GH]$ and hence,

$$F \prec t \cdot \text{supp}(F) \prec [GH] \prec A$$

contradicting our hypothesis that $A \in \text{Avoid}(m, \mathcal{F}, t-1)$. Therefore, we can conclude that $\text{supp}(F) \not\prec C_r$.

We remark next that for any configuration $F' \prec C_r$, we would have $\begin{bmatrix} 0 & 1 \end{bmatrix} \times F' \prec A$. So we define the following and justify why C_r avoids this family:

$$\mathcal{G} = \{F' : \exists F \in \mathcal{F}, F \prec \begin{bmatrix} 0 & 1 \end{bmatrix} \times F', F \not\prec \begin{bmatrix} 0 & 1 \end{bmatrix} \times F'', \forall F'' \prec F', F'' \neq F'\}.$$

To describe in words, $\mathcal{G}$ consists of all configurations F' so that there exists some $F \in \mathcal{F}$ which is a configuration of $\begin{bmatrix} 0 & 1 \end{bmatrix} \times F'$ and F' is minimal such configuration; that is, if $F'' \prec F'$ and $F \prec \begin{bmatrix} 0 & 1 \end{bmatrix} \times F''$, then it necessarily follows that $F'' = F'$. By defining $\mathcal{G}$ in such a way, we ensure that C is avoiding a minimal such family. Now, we conclude that

$$C_r \in \text{Avoid}(m-1, \{\text{supp}(F), \mathcal{G}\}, t-1).$$

Finally, using the Remark 5.7, we conclude that:

$$\begin{aligned}
\|A\| &= \|[B_r C_r D_r]\| + \|C_r\| \\
&\leq \text{forb}(m, \mathcal{F}, t-1) + \text{forb}(m-1, \{\text{supp}(F), \mathcal{G}\}, t-1) \\
&\leq \text{forb}(m, \mathcal{F}, t-1) + (t-1) \cdot \text{forb}(m-1, \{\text{supp}(F), \mathcal{G}\})[4]
\end{aligned}$$

5.2 Examples of Multiplicity Induction

We want to illustrate the usefulness of multiplicity induction and how it has been used to prove some significant results in the study of forbidden configurations.

Definition 5.9 *Let e, f, g and h be non-negative integers. We define $F_{e,f,g,h}$ as the $(e+f+g+h) \times 2$ matrix consisting of e rows of $\begin{bmatrix} 1 & 1 \end{bmatrix}$, f rows of $\begin{bmatrix} 1 & 0 \end{bmatrix}$, g rows of $\begin{bmatrix} 0 & 1 \end{bmatrix}$ and h rows of $\begin{bmatrix} 0 & 0 \end{bmatrix}$.*

Remark 5.10 *We remark that*

$$F_{e,f,g,h} = [\mathbf{1}_{e+f}\mathbf{0}_{g+h}|\mathbf{1}_e\mathbf{0}_f\mathbf{1}_g\mathbf{0}_h].$$

Recall that for a matrix F, we defined the following notation:

$$t \cdot F = \underbrace{[F|\dots|F]}_{t}.$$

We begin by stating a lemma and presenting a proof, which does not use multiplicity induction but is presented because it is quite interesting and clever.

Lemma 5.11 *[4] There exists a $c > 0$ such that forb$(m, \{F_{0,2,2,0}, 2 \cdot F_{0,1,2,0}\}) \leq c \cdot m$.*

Proof: Let $A \in \text{Avoid}(m, 2 \cdot F_{0,2,2,0})$ and without loss of generality, let $\|A\| = \text{forb}(m, \{F_{0,2,2,0}, 2 \cdot F_{0,1,2,0}\})$. Note that A is simple by hypothesis, and hence, we disregard the columns of A that are all 0's and all 1's. Let us consider the columns with column sum k, where $1 \leq k \leq m-1$. Let X_i denote the submatrix formed by the columns of A, which have column sum i. Fix i, and suppose that $\|X_i\| \geq 3$. We will argue that X_i has to be one of the following two types.

We begin by noting that X_i avoids $F_{0,2,2,0}$, or otherwise, $F_{0,2,2,0} \prec X_i \prec A$ and we obtain a contradiction to our original hypothesis. Moreover, we note that by the simplicity of A, as X_i is formed by a subset of columns of A, X_i is also simple. Let x_i be the number of columns of X_i. We denote by $\mathbf{1}_{a\times b}$, the a-rowed, b-columned matrix of all 1's. Likewise, by $\mathbf{0}_{a\times b}$, the a-rowed, b-columned matrix of all 0's. Anstee and Lu [4] have shown that after permutations of rows and columns, X_i has one of the two following structures:

$$X_i = \begin{bmatrix} I_{x_i} \\ \mathbf{1}_{(i-1)\times x_i} \\ \mathbf{0}_{(m-x_i-i+1)\times x_i} \end{bmatrix} \quad \text{or} \quad X_i = \begin{bmatrix} I^c_{x_i} \\ \mathbf{1}_{(i-x_i+1)\times x_i} \\ \mathbf{0}_{(m-i-1)\times x_i} \end{bmatrix}.$$

We remark that in either case, every column of X_i has column sum i. In the first case, which we shall call Type 1 as done by Anstee and Sali, each column has exactly one 1 contributed from I_{x_i} and $(i-1)$ 1's contributed from $\mathbf{1}_{(i-x_i+1)\times 1}$. In the second case, referred to as Type 2, $I^c_{x_i}$ contributes $(i-1)$ 1's while, $\mathbf{1}_{(i-x_i+1)\times 1}$ contributes $(i-x_i+1)$ 1's.

Denote by A_i, the first x_i rows in the above representations of X_i. Denote by B_i and C_i, the rows of 1's and 0's in the above representations of X_i, respectively. We will show some preliminary claims before proving the result.

We begin by claiming the following: if $i < j$, then we have that $B_i \subset B_j$. What does this statement mean? It means that, assuming we have $i < j$, the rows of X_i that form

B_i are necessarily contained in the rows that form B_j in X_j.

We show the above claim for Type 1 structures however, the argument works similarly for Type 2 structures as well. First, define the following set:

$$T(1) = \{i : X_i \text{ is of Type 1 and } \|X_i\| \geq 2\}.$$

Let $i, j \in T(1)$, and suppose for contradiction that there exists a row $p \in B_i \setminus B_j$. Now we know that $|B_j| > |B_i|$, and moreover, by our assumption, $B_j \setminus B_i$ has at least 2 rows, and we justify it as follows. Every column of X_i has column sum i, which is strictly less than the column sums of columns of X_j, which is j. Now if $B_j = B_i$, since $i < j$ and a row from I_{x_i} contributes 1 to the column sum i, there are two rows from I_{x_j} that contribute 1 to the sum j. But this is a contradiction since $\begin{bmatrix}1\\1\end{bmatrix} \not\prec I_{x_j}$. So we must have that $B_j \neq B_i$. Now suppose $|B_j \setminus B_i| = 1$. We observe that the row p contributes 1 to the column sum i but not to column sum j, and the row in $B_j \setminus B_i$ contributes 1 to column sum j but not column sum i. Then it follows that all the remaining columns contribute either 0 to both column sums i and j, or contribute 1 to both column sums i and j. This yields that $i = j$, contradicting our hypothesis that $i < j$. Then we conclude that $|B_j \setminus B_i| > 2$. Let $r, s \in B_j \setminus B_i$. We claim that the rows p, r and s contain a configuration that, by hypothesis, should be avoided by A. Consider the matrix $[X_i|X_j]$, formed by concatenating X_i and X_j. By hypothesis, since $i, j \in T(1)$, $\|[X_i|X_j]\| \geq 4$. Since $p \in B_i \setminus B_j$, the row p of $[X_i|X_j]$ has at least four 1's; where there are 1's in row p, row r and s strictly contribute 0's. Now, row $r, s \in B_j \setminus B_i$, so each contributes at least four 1's and here, row p strictly has 0's. To demonstrate, we have the following structure:

$$[X_i|X_j]|_{\{p,r,s\}} = \begin{bmatrix}1 & \dots & 1 & 0 & \dots & 0\\ 0 & \dots & 0 & 1 & \dots & 1\\ 0 & \dots & 0 & 1 & \dots & 1\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 0 & \dots & 1 & 0\\ 0 & 1 & \dots & 0 & 1\\ 0 & 1 & \dots & 0 & 1\end{bmatrix}}_{\geq 2 \text{ copies of } F_{0,2,1,0}}.$$

We see that $2 \cdot F_{0,2,1,0} \prec [X_i|X_j] \prec A$, and so we have a contradiction. This leads us to conclude that in fact, if $i, j \in T(1)$ with $i < j$, then $B_i \subset B_j$.

Now, let us form a matrix, Y_1 by concatenating all X_i, where $i \in T(1)$. A simple observation is that

$$\|Y\| = \sum_{i \in T(1)} \|X_i\| = \sum_{i \in T(1)} x_i = \sum_{i \in T(1)} |A_i|.$$

For contradiction, assume that $\|Y\| = \sum_{i \in T(1)} |A_i| > 3m$. Since A_i is the set of rows of X_i which contain the identity matrix, if $\|Y\| > 3m$, we have that in total we have more than $4m$ 1's in the matrix $Y|_{\{A_i\}_{i \in T(1)}}$. In particular since each row and column in

A_i has exactly one 1. Since $Y \prec A$, Y has at most m rows and so, it follows that there exists a row p and a size 4 set, say $\{i_{k_1}, i_{k_2}, i_{k_3}, i_{k_4}\}$ with so that $i_{k_1} < i_{k_2} < i_{k_3} < i_{k_4}$ so that $p \in A_{i_{k_j}}$ for all $j \in \{1, 2, 3, 4\}$. We claim then that the configuration $2 \cdot F_{0,1,2,0}$ occurs in Y, leading us to a contradiction. First, we note that $|B_{i_{k_4}} \setminus B_{i_{k_2}}| \geq 2$ as $B_{i_{k_1}} \subset B_{i_{k_2}} \subset B_{i_{k_4}}$, and so we can choose rows $r, s \in B_{i_{k_4}} \setminus B_{i_{k_2}}$. Let's consider the columns p, r, and s; notice that we can have a column from each of $A_{i_{k_1}}$ and $A_{i_{k_2}}$, which contain the lone 1, and it will have 0's in the rows r and s. Moreover, since $p \notin B_{i_{k_4}} \setminus B_{i_{k_2}}$, it will have 0's the columns where r and s have 1's. So we have found four columns, a column each from $A_{i_{k_1}}$ and $A_{i_{k_2}}$ which contain 1's in row p and 0's in rows r and s, and columns from $X_{i_{k_4}}$ which contain 1's in rows r and s and 0's in row p. This is precisely the structure $2 \cdot F_{0,1,2,0}$. Since $2 \cdot F_{0,1,2,0} \prec Y \prec A$, we get a contradiction to our original hypothesis. We conclude that indeed $\|Y\| \leq 3m$.

Through a similar procedure, we would define $T(2) = \{i : i \text{ is of Type 1 and } \|X_i\| \geq 2\}$ and form a matrix Z by concatenating all X_i where $i \in T(2)$. By the same argument as above, we obtain that $\|Z\| \leq 3m$.

Finally, we observe that

$$\|A\| = \text{forb}(m, \{F_{0,2,2,0}, 2 \cdot F_{0,1,2,0}\}) \leq \|Y\| + \|Z\| + 3(m-1) + 2,$$

where the last term comes from a possible column of all 1's and a possible column of all 0's, while the second last term represents all $i \in [m] \setminus \{0, m\}$ such that $i \notin T(1), i \notin T(2)$. We observe then that

$$\|A\| \leq 3m + 3m + 3(m-1) + 2 \leq 9m,$$

proving our claim. ■

A very similar proof gives us the more general lemma below:

Lemma 5.12 *forb*$(m, \{F_{0,2,2,0}, t \cdot F_{0,1,2,0}\})$ *is* $O(m)$.

Now we prove the theorem that requires Lemma 5.11 above.

Theorem 5.13 *[4] We have that forb*$(m, 2 \cdot F_{0,2,2,0}) = O(m^2)$.

Proof: We perform induction on the rows, m. Assume our inductive hypothesis: for $1 \leq k \leq m-1$, there exists a $c > 0$ such that we have

$$\text{forb}(k, 2 \cdot F_{0,2,2,0}) c \cdot k^2.$$

Additionally, remark that $F_{0,2,2,0} = \text{supp}(t \cdot F_{0,2,2,0})$ and the maximum column multiplicity in $t \cdot F_{0,2,2,0}$ is t. With $t = 2$, we can apply the result we obtained using Remark 6.7 and conclude:

$$\text{forb}(m, 2 \cdot F_{0,2,2,0}) \leq \text{forb}(m-1, 2 \cdot F_{0,2,2,0}) + \text{forb}(m-1, \{F_{0,2,2,0}, 2 \cdot F_{0,1,2,0}\}).$$

We have shown that the inductive children of $A \in \text{forb}(m, t \cdot F_{0,2,2,0})$ avoid $\{F_{0,2,2,0}, t \cdot F_{0,1,2,0}\}$ in the result that follows. As such, we have omitted the reasoning that if $A \in \text{Avoid}(m, 2 \cdot F_{0,2,2,0})$ then the inductive children avoid $\{F_{0,2,2,0}, 2 \cdot F_{0,1,2,0}\}$.
From Lemma 5.11, we know there exists a $c' > 0$ so that

$$forb(m-1, \{F_{0,2,2,0}, 2 \cdot F_{0,1,2,0}\}) \leq c' \cdot (m-1).$$

By induction on m, and our calculation above, we have that

$$\text{forb}(m, 2 \cdot F_{0,2,2,0}) \leq c \cdot (m-1)^2 + c' \cdot (m-1) \leq d \cdot m^2$$

where $d = \max\{c, c'\}$. ■

Theorem 5.14 *For $t \geq 2$, forb$(m, t \cdot F_{0,2,2,0}) = O(m^2)$. [4]*

Proof: By Lemma 5.13, there exists a $c' > 0$ such that $\text{forb}(m-1, \{F_{0,2,2,0}, t \cdot F_{0,1,2,0}\}) \leq c' \cdot m$. We perform induction on the rows, m. Assume that for all $1 \leq k \leq m-1$, we have

$$\text{forb}(k, t \cdot F_{0,2,2,0}, t-1) = O(k^2).$$

So, we can find a $c > c'$ such that $\text{forb}(k, t \cdot \{F_{0,2,2,0}, t-1) \leq c \cdot k^2$. Additionally, remark that $F_{0,2,2,0} = \text{supp}(t \cdot F_{0,2,2,0})$ and the maximum column multiplicity in $t \cdot F_{0,2,2,0}$ is t. Moreover, we make the following observation:

$$\begin{aligned} t \cdot F_{0,2,2,0} &= \underbrace{[F_{0,2,2,0}| \dots |F_{0,2,2,0}]}_{t} \\ &= \left[\underbrace{\begin{bmatrix} 1 & \dots & 1 \\ 1 & \dots & 1 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \end{bmatrix}}_{t} \middle| \cdots \middle| \underbrace{\begin{bmatrix} 1 & \dots & 1 \\ 1 & \dots & 1 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \end{bmatrix}}_{t} \right] = \left[t \cdot \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T \middle| t \cdot \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T \right] \end{aligned}$$

Then deleting the first row gives us $\left[t \cdot \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T | t \cdot \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T \right]$. We observe the following:

$$[0 \;\; 1] \times \left[t \cdot \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T | t \cdot \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T \right] = \left[\underbrace{\begin{bmatrix} 0 & \dots & 0 \\ 1 & \dots & 1 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \end{bmatrix}}_{t} \underbrace{\begin{bmatrix} 1 & \dots & 1 \\ 1 & \dots & 1 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \end{bmatrix}}_{t} \underbrace{\begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ 1 & \dots & 1 \\ 1 & \dots & 1 \end{bmatrix}}_{t} \underbrace{\begin{bmatrix} 1 & \dots & 1 \\ 0 & \dots & 0 \\ 1 & \dots & 1 \\ 1 & \dots & 1 \end{bmatrix}}_{t} \right]$$

and we remark that

$$F_{0,2,2,0} \prec [0 \;\; 1] \times \left[t \cdot \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T | t \cdot \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T \right]$$

since the second and third blocks of t columns are precisely $F_{0,2,2,0}$. Now, let $A \in \text{Avoid}(m, t \cdot F_{0,2,2,0}, t-1)$ with $\|A\| = \text{forb}(m, t \cdot F_{0,2,2,0}, t-1)$. Find a row r and perform the standard decomposition as described for $(t-1)$ – simple $(0,1)$-matrices. Then by our remark, we have the following result:

$$\begin{aligned}\|A\| &= \|[BCD]\| + \|C\| \\ &\leq \text{forb}(m-1, t \cdot F_{0,2,2,0}, t-1) + \text{forb}(m-1, \{F_{0,2,2,0}, t \cdot F_{0,1,2,0}\}) \\ &\leq c \cdot (m-1)^2 + c' \cdot (m-1) \\ &\leq c \cdot m^2.\end{aligned}$$

So we conclude that $\text{forb}(k, t \cdot F_{0,2,2,0}, t-1) \leq c \cdot m^2$, which gives us

$$\text{forb}(k, t \cdot F_{0,2,2,0}) \leq \text{forb}(k, t \cdot F_{0,2,2,0}, t-1) \leq t \cdot \text{forb}(k, t \cdot F_{0,2,2,0}) \leq c \cdot tm^2. \quad \blacksquare$$

In fact, the above results can be generalized. We looked at the asymptotics of forbidding $t \cdot F_{0,2,2,0}$; naturally, we would try and extend this to find the asymptotics of forbidding $t \cdot F_{0,k,k,0}$. Anstee and Lu[4] have found the following results, generalizing the above lemma and theorem for a general k.

Lemma 5.15 *For $t \geq 2$, forb$(m, \{F_{0,k,k,0}, t \cdot F_{0,k-1,k,0})= O(m^{k-1})$.*

Theorem 5.16 *For $t \geq 2$, forb$(m, t \cdot F_{0,k,k,0}) = O(m^k)$. [4]*

The following is another application of multiplicity induction.

Theorem 5.17 *[4] Suppose that F is a simple $(0,1)$-matrix and forb$(m, F) = O(m^\ell)$. Then it follows that for any $t \in \mathbb{N}, t \geq 2$ we have forb$(m, t \cdot F) = O(m^{\ell+1})$.*

Proof: We assume the inductive hypothesis for all $1 \leq k \leq m-1$, that $\text{forb}(k, t \cdot F) = O(m^{\ell+1})$. As before, define

$$\mathcal{G} = \{F' : F \prec \begin{bmatrix}0 & 1\end{bmatrix} \times F' \text{ and } F \not\prec \begin{bmatrix}0 & 1\end{bmatrix} \times F'' \text{ for all } F'' \prec F', F'' \neq F'\}.$$

As in the above subsection, where we explained the process of standard decomposition for t – simple matrices, we observe that $F' \prec F$. In particular, $\mathcal{G} \prec F$.

Then, from our discussion above and induction on m, we have that:

$$\begin{aligned}\text{forb}(m, t \cdot F) &\leq \text{forb}(m, t \cdot F, t-1) \\ &\leq \text{forb}(m-1, t \cdot F, t-1) + (t-1) \cdot \text{forb}(m-1, \{\mathcal{G} \cup F\})\end{aligned}$$

We would like to show that $\text{forb}(m, t \cdot F) \leq c \cdot m^{\ell+1}$ for some constant $c > 0$. We know from our inductive hypothesis that there exists a constant c' such that

$$\text{forb}(m-1, \{\mathcal{G} \cup F\}) \leq c' \cdot (m-1)^\ell.$$

By induction on m, we know that for some $c > 0$,

$$\text{forb}(m-1, t \cdot F) \le c \cdot (m-1)^{\ell+1}.$$

Putting it all together, we obtain that

$$\text{forb}(m, t \cdot F) \le c' \cdot (m-1)^{\ell} + c \cdot (m-1)^{\ell+1} \le c \cdot m^{\ell+1}.$$

and that completes the proof of our claim. ■

Recall the Sauer-Perles-Shelah bound for K_k, which is $\text{forb}(m, K_k) = O(m^{k-1})$. Using this bound and the above theorem we have proved, we get a natural result as our last example of how multiplicity induction can be useful.

Theorem 5.18 *[4] If F is any k-rowed matrix, then forb$(m, F) = O(m^k)$.*

Proof: First, we remark that K_k is the maximal simple $(0,1)$-matrix on k rows. Hence, for any k-rowed matrix F, we know that $\text{supp}(F) \prec K_k$ and hence, if F has maximum column multiplicity t, $F \prec t \cdot \text{supp}(F) \prec t \cdot K_k$. Therefore,

$$\text{forb}(m, F) \le \text{forb}(m, t \cdot \text{supp}(F)) \le \text{forb}(m, t \cdot K_k).$$

Since $\text{forb}(m, K_k)$ is $O(m^{k-1})$ then by Theorem 5.17, $\text{forb}(m, t \cdot K_k)$ is $O(m^k)$. ■

6 Sporadic Induction

There are other methods of induction that have proven useful in the study of forbidden configurations. Below we state another induction idea as a theorem and give examples.

We begin by stating a general remark.

Remark 6.1 *Suppose $f : \mathbb{N} \to \mathbb{R}$ is an arithmetic function, with $f(m) = O(m^\alpha)$ for some $\alpha \ge 0$. Then, it is necessarily true that*

$$\sum_{i=0}^{m} f(i) = O(m^{\alpha+1}).$$

Using the above remark, we state and prove the theorem below:

Theorem 6.2 *[1] Suppose that G is a k-rowed, ℓ-columned $(0,1)$-matrix, with forb$(m, G) = O(m^\alpha)$, for some $\alpha \ge 0$. Suppose F is of the following form:*

$$F = \begin{bmatrix} 0 & \dots & 0 \\ 1 & \dots & 1 \\ & G & \end{bmatrix}.$$

Then, forb$(m, F) = O(m^{\alpha+1})$.

Proof: The proof is due to Anstee and Sali. We begin by taking $A \in \text{Avoid}(m, F)$ with $\|A\| = \text{forb}(m, F)$. Assume the following inductive hypothesis: for all $3 \leq n \leq m-1$, we have that $\text{forb}(n, F) = O(n^{\alpha+1})$. Next, let us denote by Z_i the set of columns of A with the first $i+1$ rows of the form $\mathbf{1}_i\mathbf{0}_1$, and with J_i, denote the set of columns of A with the first $i+1$ rows of the form $\mathbf{0}_i\mathbf{1}_1$. This has potentially not considered the columns of 1's and the columns of 0's if they are present in A.

Now we appeal to the inductive hypothesis: note that if $G \prec Z_i$ or $G \prec J_i$ for any i, we have that $F \prec A$, violating the assumption on A. Therefore, for all i, $Z_i \in \text{Avoid}(m-i-1, G)$ and $J_i \in \text{Avoid}(m-i-1, G)$. Hence, $\|Z_i\| \leq \text{forb}(m-i-1, G)$ and $\|J_i\| \leq \text{forb}(m-i-1, G)$. Since for all i, $\text{forb}(m-i-1, G) = O((m-i-1)^{\alpha})$, summing over all $1 \leq i \leq m-1$ gives us:

$$\|A\| \leq 2\sum_{i=1}^{m-1} \text{forb}(m-i-1, G) + 2 = O(m^{\alpha+1}),$$

where $O(m^{\alpha+1})$ is obtained from Remark 6.1 that we stated and did not prove. ■

Now we present an example where this sporadic induction, one which appears rarely, but can be useful. Let

$$F = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ G \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We will show that $\text{forb}(m, F) = O(m^2)$, once using the sporadic induction above, and using counting methods to prove it a second way, in order to confirm that the sporadic induction does actually work.

We first show that $\text{forb}(m, G) = O(m)$. Let $A \in \text{Avoid}(m, G)$, with $\|A\| = \text{forb}(m, G)$. We will count the maximum number of columns A can have before the configuration G shows up. A can have a column of 0's and 1's. Next, any other column of A must have at least one 1, and one 0. But observe that none of these columns can have two copies of 0, or otherwise we have the existence of some row permutation of configuration G. So we count how many distinct ways we can place exactly one 0 in m different rows, which is precisely $\binom{m}{1} = m$ different ways. So we have that A can have at most $m+2$ columns to avoid G. Therefore, $\text{forb}(m, G) = O(m)$.

By the sporadic induction theorem above, we conclude that $\text{forb}(m, F) = O(m^2)$. Now let us show another way that $\text{forb}(m, F) = O(m^2)$, so that we can see that the two answers coincide. If $A \in \text{Avoid}(m, F)$ with $\|A\| = \text{forb}(m, F)$, we count what the maximum number of distinct columns A can have. Disregarding the columns with all

1's and 0's, we look at how many other columns A can have. Every additional column has to have at least one 1 and one 0. To avoid G, we must make sure that there are at most two 0's or at most one 1. There are $\binom{m}{2} + \binom{m}{1}$ distinct columns with either one 0 or two 0's. There are $\binom{m}{1}$ distinct columns with exactly one 1. So in total we have that

$$\|A\| = \binom{m}{2} + 2 \cdot \binom{m}{1} + 2.$$

And so $\|A\|$ is $O(m^2)$.

We have remarked above that this sporadic induction may not be useful in all situations. The following example is one where we show that sporadic induction will give an asymptotic bound that is correct, yet not sharp enough. Recall the matrix $G_{6\times3}$:

$$G_{6\times3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Deleting the first two rows of 0's and 1's, we yield a matrix G:

$$G = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We note that the last two columns of G is $F_{0,2,2,0}$, a configuration explored in the section on multiplicity induction. Anstee and Sali have shown that if F is a configuration such that $F_{0,2,2,0} \prec F$, then $\text{forb}(m, F) = \Theta(m^2)$. Therefore, we can conclude that $\text{forb}(m, G) = \Theta(m^2)$. By the sporadic induction above, since $\text{forb}(m, G) = \Theta(m^2)$, and in particular, $\text{forb}(m, G) = O(m^2)$, we conclude that $\text{forb}(m, G_{6\times3}) = O(m^3)$. Recall that we have already noted that $\text{forb}(m, G_{6\times3})$ is $O(m^2)$. We observe that the sporadic induction above, yielded $O(m^3)$; this is a correct but not a sharp bound. This demonstrates the limitations of using induction in the study of certain forbidden configurations.

7 Conclusion

We observed how useful and multi-faceted induction can be, when facing problems in forbidden configurations. An important result in extremal theory, the Shattered Set Lemma, was proven using the simplest of induction which we called basic induction. The Sauer bound was cleverly proven using the method of standard induction. This result can itself be used inductively, such as in the section on repeated induction. Lastly, we saw how the results applied to non-simple $(t-1)$-simple matrices were useful in determining bounds of matrices which were simple.

At the same time, we saw that induction may fall short at times, yielding bounds which may be correct but not sharp. This was observed in particular, in Section 6. This emphasizes the fact that induction, although ubiquitous throughout the study of forbidden configurations, has its drawbacks. It is undoubtedly in the interest to the curious mathematician to develop other clever induction methods (or indeed other methods) which may be of use, if the first attempt with induction fails.

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