Math 267, Section 202 : HW 4

All **five** questions are due **Wednesday, January 30th**.

1. Consider the periodic signal, period $\mathtt{T} = 4$, given by:

$$g(x) = \begin{cases} 0 & \text{for } -2 < x < -1 \\ x & \text{for } -1 < x < 1 \\ 0 & \text{for } 1 < x < 2 \end{cases}$$

(and repeated periodically.)

Compute the Fourier coefficients c_k of $g(x)$. Write out the Fourier series.

Answer

$$\begin{aligned} c_k &= \frac{1}{\mathtt{T}} \int_a^{a+\mathtt{T}} g(x)\, e^{-i\,k\,\frac{2\pi}{\mathtt{T}}x} \\ &= \frac{1}{4} \int_{-1}^{1} x\, e^{-i\,k\,\frac{\pi}{2}x} \\ &= \frac{1}{4} x \left. \frac{e^{-i\,k\,\frac{\pi}{2}x}}{-i\,k\,\frac{\pi}{2}} \right|_{x=-1}^{1} - \frac{1}{4} \int_{-1}^{1} \frac{e^{-i\,k\,\frac{\pi}{2}x}}{-i\,k\,\frac{\pi}{2}} \\ &= \frac{1}{4} \left(\frac{x}{-i\,k\,\frac{\pi}{2}} - \frac{1}{\left(-i\,k\,\frac{\pi}{2}\right)^2} \right) \left. e^{-i\,k\,\frac{\pi}{2}x} \right|_{x=-1}^{1} \end{aligned}$$

There are several ways to simplify this using $e^{-i\,k\,\frac{\pi}{2}(\pm 1)} = \left(e^{i\frac{\pi}{2}}\right)^{\mp k} = i^{\mp k}$, and $i^{-k} = (-1)^k\, i^k$. In particular:

$$c_k = \frac{i^{k+1}}{2\,k\pi} \left((-1)^k + 1\right) + \frac{i^k}{k^2\pi^2} \left((-1)^k - 1\right)$$

2. Let $f(x)$ be a 2-periodic function (i.e. period $T = 2$) and

$$f(x) = \begin{cases} 0 & \text{for } 100 < x < 101 \\ x & \text{for } 101 \le x < 102 \end{cases}$$

   (a) What is the value of $f(301.5)$?

   (b) Compute the Fourier coefficients c_k of $f(x)$.

Answer

   (a) Since the period is $\mathtt{T} = 2$, $f(x) = f(x - k * 2)$ whenever k is an integer.

$$f(301.5) = f(301.5 - 100 * 2) = f(101.5) = 101.5$$

(b)

$$\begin{aligned} c_k &= \frac{1}{\mathtt{T}} \int_a^{a+\mathtt{T}} f(x)\, e^{-i\,k\,\frac{2\pi}{\mathtt{T}}\,x} \\ &= \frac{1}{2} \int_{100}^{102} f(x)\, e^{-i\,k\,\pi x} \\ &= 0 + \frac{1}{2} \int_{101}^{102} x\, e^{-i\,k\,\pi x} \\ &= \frac{1}{2} \left(x \frac{e^{-i\,k\,\pi x}}{-i\,k\pi} - \frac{e^{-i\,k\,\pi x}}{\left(-i\,k\pi\right)^2} \right) \Bigg|_{x=101}^{102} \end{aligned}$$

To simplify, we will use $e^{-i\,k\pi 101} = (-1)^k$ and $e^{-i\,k\pi 102} = 1$, since $102 \cdot k$ is even and $101 \cdot k$ is odd if and only if k is odd.

$$c_k = \frac{i}{2k\pi} \left(102 - (-1)^k \cdot 101\right) + \frac{1}{2k^2\pi^2} \left(1 - (-1)^k\right)$$

3. Consider a pair of periodic signals $f(t)$ and $g(t)$, both with period $\mathtt{T} = 2\pi$. Suppose that all we know is that the Fourier coefficients c_k of $f(t)$ and the Fourier coefficients d_k of $g(t)$ satisfy:

$$\begin{cases} c_k = d_k & \text{for } -100 \leq k \leq 100 \text{ ,} \\ c_k = d_k + 3^{-|k|} & \text{for } |k| > 100. \end{cases}$$

Compute $\int_{-\pi}^{\pi} |f(t) - g(t)|^2 dt$.

Answer

Since $f(t)$ and $g(t)$ both have period $\mathtt{T} = 2\pi$, then so does $h(t) = f(t) - g(t)$. Moreover:

$$\left.\begin{aligned} f(t) &= \sum_k c_k e^{i\,k\,\frac{2\pi}{\mathtt{T}}\,t} \\ g(t) &= \sum_k d_k e^{i\,k\,\frac{2\pi}{\mathtt{T}}\,t} \end{aligned}\right\} \Rightarrow\ f(t) - g(t) = h(t) = \sum_k \left(c_k - d_k\right) e^{i\,k\,\frac{2\pi}{\mathtt{T}}\,t}$$

Now forget for a minute where $h(t)$ comes from - it has period 2π, so by Parseval's theorem (or *orthogonality*):

$$\frac{1}{2\pi} \int_{-\pi}^{-\pi+2\pi} |h(t)|^2\, dt = \sum_k |h_k|^2$$

where h_k are the Fourier series coefficients for $h(t)$. We found above that,

$$h_k = c_k - d_k = \begin{cases} 0 & \text{for } -100 \leq k \leq 100 \\ -\,3^{-|k|} & \text{for } |k| > 100 \end{cases}$$

This means:

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} |h(t)|^2\,dt = \sum_{k=-\infty}^{-101} \left|-3^{-|k|}\right|^2 + \sum_{k=-100}^{100} 0 + \sum_{k=101}^{\infty} \left|-3^{-|k|}\right|^2$$

$$= 2\sum_{k=101}^{\infty} 3^{-2|k|}$$

$$= 2\left(\sum_{k=0}^{\infty}\left(\frac{1}{9}\right)^k - \sum_{k=0}^{100}\left(\frac{1}{9}\right)^k\right)$$

$$= 2\left(\frac{1-\left(\frac{1}{9}\right)^{\infty}}{1-\frac{1}{9}} - \frac{1-\left(\frac{1}{9}\right)^{101}}{1-\frac{1}{9}}\right)$$

Don't forget about $\frac{1}{\mathtt{T}}$ from the left side. Conclude that:

$$\int_{-\pi}^{\pi} |f(t)-g(t)|^2\,dt = \int_{-\pi}^{+\pi} |h(t)|^2\,dt = \frac{\pi}{2\cdot 9^{100}}$$

4. Consider the function, defined for $0 < x < 2$ by:

$$f(x) = \begin{cases} 0 & \text{for } 0 < x < 1 \\ 1 & \text{for } 1 < x < 2 \end{cases}$$

(a) Sketch the graph of *even extension*, $f_{even}(x)$.

(b) Compute the Fourier coefficients c_k of the *even extension*.

(c) What is the sum of the resulting Fourier series for $x = 2$? Give a numeric value.

Answer

(a) The even extension is $+1$ for $-2 < x < -1$, zero for $-1 < x < 1$, then $+1$ for $1 < x < 2$. This pattern is repeated.

The even extension has period $\mathtt{T} = 4$. It is convenient to think of a period as starting at $a = -1$, because then the first cycle is easy to describe: zero for $-1 < x < 1$ and $+1$ for $1 < x < 3$.

(b)

$$\begin{aligned}
c_k &= \frac{1}{\mathtt{T}}\int_a^{a+\mathtt{T}} f_{even}(x)\,\overline{e^{i\,k\,\frac{2\pi}{\mathtt{T}}x}} \\
&= \frac{1}{4}\int_{-1}^{-1+4} f_{even}(x)\,e^{-i\,k\,\frac{2\pi}{4}x} \\
&= 0 + \frac{1}{4}\int_1^3 1\cdot\, e^{-i\,k\,\frac{\pi}{2}x} \\
&= \frac{i}{2\,k\pi}\left(e^{-i\,3k\,\frac{\pi}{2}} - e^{-i\,k\,\frac{\pi}{2}}\right) \\
&= \frac{i}{2\,k\pi}e^{-i\,k\pi}\left(e^{-i\,k\,\frac{\pi}{2}} - e^{+i\,k\,\frac{\pi}{2}}\right) \\
&= \frac{i^{k+1}\,(-1)^k\left((-1)^k - 1\right)}{2\,k\,\pi} \\
c_0 &= \frac{1}{\mathtt{T}}\int_a^{a+\mathtt{T}} f_{even}(x)\cdot 1 \\
&= 0 + \frac{1}{4}\int_1^3 1\cdot 1 \\
&= \frac{1}{2}
\end{aligned}$$

Don't forget to calculate c_0 separately! It is not necessary to simplify c_k to the level shown.

5. For $-1 < x < 1$,

$$x^3 = \sum_{k\neq 0} i\left(\frac{1}{k\pi} - \frac{3}{2}\frac{1}{k^3\pi^3}\right)e^{i\,k\,\pi\,x}.$$

Rewrite this as a *real Fourier series.*

Answer

Use Euler: $e^{ik\pi x} = \cos(k\pi x) + i\sin(k\pi x)$, and split the sum:

$$x^3 = \sum_{m=-\infty}^{-1} i\left(\frac{1}{m\pi} - \frac{3}{2}\frac{1}{m^3\pi^3}\right)(\cos(m\pi x) + i\sin(m\pi x)) + \sum_{k=1}^{\infty} i\left(\frac{1}{k\pi} - \frac{3}{2}\frac{1}{k^3\pi^3}\right)(\cos(k\pi x) + i\sin(k\pi x))$$

Now flip the sign of m, and match like terms. In particular, note that $\cos(-k\pi x) = \cos(k\pi x)$ and $\sin(-k\pi x) = -\sin(k\pi x)$.

$$\begin{aligned}
x^3 =& \sum_{k=1}^{\infty}\left(i\left(\frac{1}{-k\pi} - \frac{3}{2}\frac{1}{(-k)^3\pi^3}\right) + i\left(\frac{1}{k\pi} - \frac{3}{2}\frac{1}{k^3\pi^3}\right)\right)\cos(k\pi x) \\
&+ \sum_{k=1}^{\infty}\left(-i\left(\frac{1}{-k\pi} - \frac{3}{2}\frac{1}{(-k)^3\pi^3}\right) + i\left(\frac{1}{k\pi} - \frac{3}{2}\frac{1}{k^3\pi^3}\right)\right) i\,\sin(k\pi x) \\
=& \sum_{k=1}^{\infty}(-2)\left(\frac{1}{k\pi} - \frac{3}{2}\frac{1}{k^3\pi^3}\right)\sin(k\pi x)
\end{aligned}$$