# Math 267 - Midterm 2

**7-8pm, March $7^{st}$ 2013**

**Last Name:**

**First Name:**

**SID:**

**Instructor:** **Section:**

## Instructions

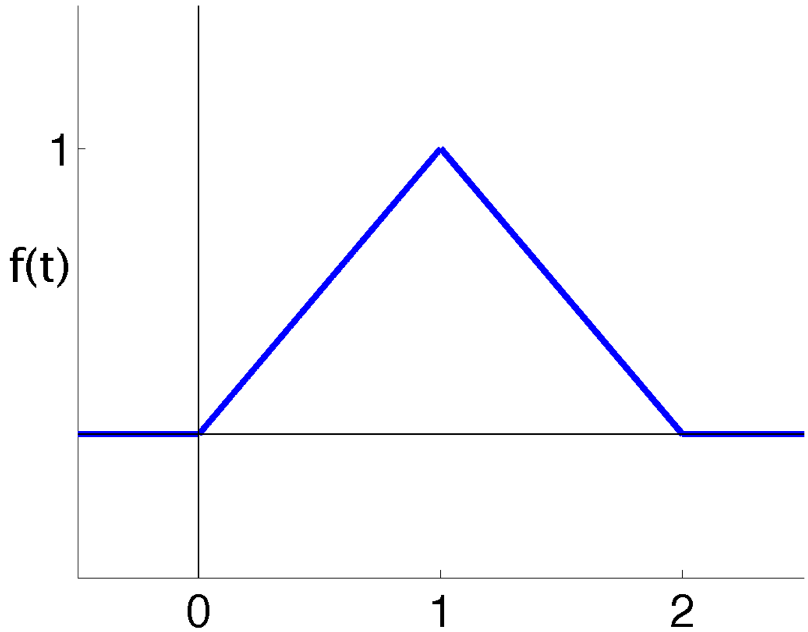
- **Do not open** this test until instructed to do so.
- This exam should have **7** pages. The last page is a formula sheet.
- Use the reverse side of each page if you need extra space.
- Show all your work and write answers in the spaces indicated. A correct answer without intermediate steps will receive no credit.
- Calculators, phones, textbooks and notes are **not** allowed.
- The total time allowed is 60 minutes.

| Part | Out of | Score |
|---|---|---|
| **I.** | 17 | |
| **II.** | 8 | |
| **TOTAL** | 25 | |

## Part I. [17 points]

(A) (4 points)

Compute the Fourier transform of $f(t)$, where:




**Solution** Notice that $\frac{d}{dt}f(t) = \text{rect}(t-1/2) - \text{rect}(t-3/2)$. Since $\mathcal{F}\left[\frac{d}{dt}f(t)\right] = i\omega\mathcal{F}\left[f(t)\right]$, for $\omega \neq 0$,

$$\begin{aligned}
\mathcal{F}[f(t)](\omega) &= \frac{1}{i\omega}\mathcal{F}\left[\text{rect}(t-1/2) - \text{rect}(t-3/2)\right](\omega) \\
&= \frac{1}{i\omega}\left(e^{-i\omega/2}\text{sinc}(\omega/2) - e^{-i3\omega/2}\text{sinc}(\omega/2)\right) \\
&= \frac{1}{i\omega}(e^{-i\omega}(e^{i\omega/2} - e^{-i\omega/2})\text{sinc}(\omega/2) \\
&= e^{-i\omega}\text{sinc}(\omega/2)^2
\end{aligned}$$

For $\omega = 0$, $\mathcal{F}(f(t)](0) = \int_{-\infty}^{\infty} f(t)dt = 1$.

Answer:
$e^{-i\omega}\text{sinc}(\omega/2)^2$.

(B) (4 points)

Let $f(t) = t$ and $g(t) = \text{rect}(t)$. Calculate $(f * g)(t)$.

**Solution**:

$$\begin{aligned}
(f * g)(t) = (g * f)(t) &= \int_{-\infty}^{\infty} (t - s)\,\text{rect}(s) ds \\
&= \int_{-1/2}^{1/2} (t - s) ds \\
&= \int_{-1/2}^{1/2} t ds - \int_{-1/2}^{1/2} s ds \\
&= t \int_{-1/2}^{1/2} 1 ds - \int_{-1/2}^{1/2} s ds \\
&= t.
\end{aligned}$$

Answer:
$t$

(C) (4 points)

Compute the Fourier transform of $h(t) = \cos(-3t)\sin(2t)$.

**Solution** Note that

$$\cos(-3t)\sin(2t) = \frac{1}{2}(e^{-i3t} + e^{i3t})\frac{1}{2i}(e^{i2t} - e^{-i2t})$$
$$= \frac{1}{4i}\left(e^{-it} - e^{-i5t} + e^{i5t} - e^{it}\right)$$

Thus, its Fourier transform is $\frac{2\pi}{4i}\left(\delta(t+1) - \delta(t+5) + \delta(t-5) - \delta(t-1)\right)$

Answer:
$\frac{\pi}{2i}\left(\delta(t+1) - \delta(t+5) + \delta(t-5) - \delta(t-1)\right)$.

(D) (5 points)

Consider the ODE:

$$y''(t) - y(t) = x(t)$$

**Using the Fourier transform**, find $y(t)$ when $x(t) = \delta(t)$.

**Solution**: Take Fourier transform of the differential equation to get

$$(i\omega)^2 \widehat{y}(\omega) - \widehat{y}(\omega) = \widehat{x}(\omega).$$

Since $\mathcal{F}\left[\delta(t)\right](\omega) = 1$,

$$\begin{aligned} \widehat{y}(\omega) &= \frac{1}{(i\omega)^2 - 1} \\ &= \frac{1}{2}\left(\frac{1}{i\omega - 1} - \frac{1}{i\omega + 1}\right) \end{aligned}$$

Thus, from Fourier inversion,

$$y(t) = \frac{1}{2}\left(-e^{t}u(-t) - e^{-t}u(t)\right).$$

Answer:
$y(t) = \frac{1}{2}\left(-e^{t}u(-t) - e^{-t}u(t)\right).$

**Part II. [8 points]:**

(A) (4 points)

Find $g(t)$ if $\widehat{g}(\omega) = i\,\omega\,\mathrm{rect}(\omega)$.

**Solution** Differentiation property implies

$$i\omega\mathrm{rect}(\omega) = \mathcal{F}\left[\frac{d}{dt}f(t)\right](\omega)$$

where $f(t)$ is the Fourier inversion of $\mathrm{rect}(\omega)$. From duality, $f(t) = \frac{1}{2\pi}\mathrm{sinc}(t/2)$. Thus,

$$g(t) = \frac{d}{dt}\frac{1}{2\pi}\mathrm{sinc}(t/2)$$

Here, for $t \neq 0$,

$$\begin{aligned} \frac{d}{dt}\mathrm{sinc}(t) &= \frac{d}{dt}\frac{2}{t}\sin(t/2) \\ &= \left(\frac{-2}{t^2}\sin(t/2) + \frac{2}{t}\cdot\frac{1}{2}\cos(t/2)\right) \end{aligned}$$

Note that for $t = 0$, $\frac{d}{dt}\mathrm{sinc}(t)\Big|_{t=0} = 0$. Thus,

$$g(t) = \begin{cases} \frac{1}{\pi}\left(\frac{-1}{t^2}\sin(t/2) + \frac{1}{2t}\cos(t/2)\right) & t \neq 0, \\ 0 & t = 0. \end{cases}$$

Answer:

$$g(t) = \begin{cases} \frac{1}{\pi}\left(\frac{-1}{t^2}\sin(t/2) + \frac{1}{2t}\cos(t/2)\right) & t \neq 0, \\ 0 & t = 0. \end{cases}$$

(B) (4 points)

Suppose $\widehat{y}(\omega) = \widehat{H}(\omega)\widehat{x}(\omega)$. Let

$$\widehat{H}(\omega) = \frac{1}{\omega} \quad \text{and} \quad x(t) = e^{it}u(t).$$

Find $y(t)$. (Hint: start by finding $\widehat{x}(\omega)$.)

**Solution**: Note that $\widehat{u}(\omega) = \frac{1}{i\omega} + \pi\delta(\omega)$. Thus, time shifting gives,

$$\begin{aligned}
\widehat{H}(\omega)\widehat{x}(\omega) &= \frac{1}{\omega}\left(\frac{1}{i(\omega-1)} + \pi\delta(\omega-1)\right) \\
&= \frac{1}{\omega}\frac{1}{i(\omega-1)} + \pi\frac{1}{\omega}\delta(\omega-1) \\
&= \frac{1}{\omega}\frac{1}{i(\omega-1)} + \pi\delta(\omega-1) \\
&= \frac{1}{i}\left(\frac{1}{\omega-1} - \frac{1}{\omega}\right) + \pi\delta(\omega-1) \\
&= -\frac{1}{i\omega} + \frac{1}{i(\omega-1)} + \pi\delta(\omega-1) \\
&= -\left[\frac{1}{i\omega} + \pi\delta(\omega)\right] + \pi\delta(\omega) + \left[\frac{1}{i(\omega-1)} + \pi\delta(\omega-1)\right]
\end{aligned}$$

Thus, Fourier inversion gives,

$$y(t) = -u(t) + \frac{1}{2} + e^{it}u(t).$$

Answer:
$-u(t) + \frac{1}{2} + e^{it}u(t)$.

## End of Exam