Part I. [14 points]

(A) (4 points) Let functions $g(x)$ and $h(x)$ have period 2, and complex Fourier series, $g(x) = \sum_{k=-\infty}^{\infty} c_k e^{ik\pi x}$, and $h(x) = \sum_{k=-\infty}^{\infty} d_k e^{ik\pi x}$, where $c_k = d_k + 2^{-|k|/2}$ for all integer k.

Compute $\frac{1}{2}\int_{-1}^{1} |g(x) - h(x)|^2 dx$.

$$g(x)-h(x) = \sum_{k=-\infty}^{+\infty} (c_k - d_k) e^{ik\pi x} = \sum_{k=-\infty}^{\infty} -2^{-\frac{|k|}{2}} e^{ik\frac{2\pi}{2}x}$$

call this $f(x)$

Parseval theorem:

$$\frac{1}{T}\int_a^{a+T} |f(x)|^2 = \sum_{k=-\infty}^{+\infty} |f_k|^2$$

$$\frac{1}{2}\int_{-1}^{-1+2} |g(x)-h(x)|^2 = \sum_{k=-\infty}^{+\infty} \left|-2^{-\frac{|k|}{2}}\right|^2 = \sum_{k=-\infty}^{-1} 2^{-|k|} + \sum_{k=0}^{\infty} 2^{-k}$$

$$= 2\left(\sum_{k=0}^{\infty} 2^{-k}\right) - 2^0$$

geomet. series

$$= 2\,\frac{1}{1-\frac{1}{2}} - 1$$

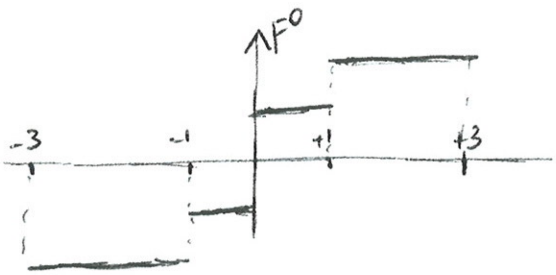
Answer: 3

(B) (2 points) Let $F^o(x)$ be the *odd* periodic extension of,

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1\,, \\ 2 & \text{for } 1 < x < 3. \end{cases}$$

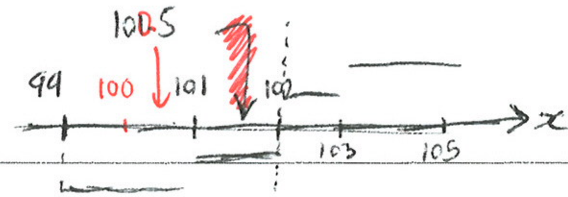
What is the value of $F^o(100.5)$?

F^o has period $T = 6$



← – this identical pattern starts at $x = 6k+3$, for integers

note: $6\times16+3 = 99 < 100.5 < 6\times17+3$



Answer: $F^o(100.5) = -2$

(C) (3 points) Compute $\int_{-1}^{1} \cos(\pi x) e^{i\pi x}\, dx$.

Fastest to use orthogonality

$$\int_{-1}^{1} \frac{e^{i\pi x} + e^{-i\pi x}}{2} \, \overline{e^{i(-1)\pi x}} \, dx$$

zero

$\frac{1}{2} \times T = \frac{1}{2} \times 2$

Answer: 1

(D) (5 points) Find all real numbers γ such that the following boundary value problem has a **nontrivial solution** (i.e. nonzero).

$$\begin{cases} X''(x) + \gamma X(x) = 0 & \text{for } 0 < x < 1, \\ X'(0) = 0, \quad X'(1) = 0. \end{cases}$$

Notice the derivatives in the boundary conditions.
You must clearly justify your answer.

Case ①: $\gamma > 0$

$$X(x) = c_1 \cos(\sqrt{\gamma} x) + c_2 \sin(\sqrt{\gamma} x)$$

$0 = X'(0) = c_2 \cos(0) \cdot \sqrt{\gamma} \implies c_2 = 0$

$0 = X'(1) = -c_1 \sin(\sqrt{\gamma}) \sqrt{\gamma} + 0 \implies \sin(\sqrt{\gamma}) = 0$ or trivial

$\Downarrow$

$\sqrt{\gamma} = k\pi$ for integer $k > 0$ (else case ②)

$\gamma = k^2 \pi^2$

Case ② $\gamma = 0$

$X(x) = c_1 + c_2 x$

$0 = X'(0) = c_2 \implies c_2 = 0$

$0 = X'(0) = 0 \implies$ true. Can choose c_1 to be anything

eg: $X(x) = 1$, which is nontrivial.

$\gamma = 0$ works.

Case ③ $\gamma < 0$

$X(x) = c_1 e^{\sqrt{-\gamma} x} + c_2 e^{-\sqrt{-\gamma} x}$

$0 = X'(0) = \sqrt{-\gamma}(c_1 - c_2) e^0 \implies c_1 = c_2$

$0 = X'(1) = c_1 \sqrt{-\gamma} \left(e^{\sqrt{-\gamma}} - e^{-\sqrt{-\gamma}}\right) \implies c_1 = 0$, since $\cancel{e^{(\text{Number})} \neq e^{-(\text{Same Number})}}$

$e^a \neq e^{-a}$ for $a \neq 0$.

Answer: $\gamma = k^2 \pi^2$ for $k \geq 0$ integer

Part II. [11 points]:
Consider the following **heat equation** with **inhomogenous** boundary conditions:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) & \text{for } 0 < x < 1,\ t > 0, \\ u(0,t) = 5, \quad u(1,t) = 5, & \text{for } t > 0. \end{cases}$$

(A) (3 points) Find a steady state solution.

seek $u(x,t) = u_p(x)$
$\Rightarrow$
$$\begin{cases} \partial_t [u_p(x)] = \partial_{xx}[u_p(x)] \\ u_p(0) = 5 = u_p(1) \end{cases} \Rightarrow \quad 0 = u_p''(x) \Rightarrow u_p(x) = A + Bx$$

$u_p(0) = 5 = u_p(1) \Rightarrow A = 5,\ B = 0$

Answer: $u_p(x) = 5$

(B) (4 points) Write the steady state as either:

$$\sum_{k=1}^{\infty} b_k \sin(k\pi x) \quad \text{or} \quad \sum_{k=1}^{\infty} c_k \left(e^{i k\pi x} - e^{-i k\pi x}\right)$$

Calculate coefficients b_k, or coefficients c_k, as you prefer.

seek $5 = \sum_{k=1}^{\infty} c_k \left(e^{ik\pi x} - e^{-ik\pi x}\right)$ for $0 < x < 1$ $\Rightarrow$ formula sheet

$$c_k = \frac{1}{2 \cdot 1} \int_0^1 5 \overline{\left(e^{ik\pi x} - e^{-ik\pi x}\right)} = \frac{5}{2} \frac{-1}{ik\pi} \left(e^{-ik\pi x} + e^{+ik\pi x}\right)\Big|_{x=0}^{1}$$

or

$$5 = \sum_{k=1}^{\infty} b_k \sin(k\pi x)$$

$$b_k = \frac{2}{1} \int_0^1 5 \sin(k\pi x)\,dx$$

$$= 10 \cdot \left[\frac{-\cos(k\pi x)}{k\pi}\right]_0^1 \quad \leftarrow k \geq 1$$

$$= \frac{10}{k\pi} \cdot \left[1 - (-1)^k\right]$$

Answer: $c_k = i \frac{5}{k\pi}\left((-1)^k - 1\right)$ or

$b_k = -\frac{10}{k\pi}\left((-1)^k - 1\right)$

(C) (4 points) Find the solution $u(x,t)$ of the given heat equation that satisfies:

$$u(x,0) = \sum_{k=1}^{100} 2^{-k} \sin(k\pi x) \quad \text{for } 0 < x < 1.$$

Use your answers of parts (A) and (B). You are **not** asked to explain separation of variables. It is acceptable here to simply write the correct answer.

homogeneous eqn.

$$\begin{cases} v_t = v_{xx} \\ v(0,t) = 0 = v(1,t) \end{cases}$$

$$u(x,t) = v(x,t) + u_p(x).$$

IC for homogeneous sol. $v(x,t)$

$\therefore$

$$v(x,0) = u(x,0) - u_p(x)$$

(A) $\rightarrow$ $= -5 + \sum_{k=1}^{100} 2^{-k} \sin(k\pi x).$

(B) $\rightarrow$ $= -\sum_{k=1}^{\infty} \frac{10}{k\pi}(1-(-1)^k) \sin(k\pi x) + \sum_{k=1}^{100} 2^{-k} \sin(k\pi x)$

So $v(x,t) = -\sum_{k=1}^{\infty} \frac{10}{k\pi}(1-(-1)^k) \sin(k\pi x) e^{-k^2\pi^2 t}$

$+ \sum_{k=1}^{100} 2^{-k} \sin(k\pi x) e^{-k^2\pi^2 t}$

used ~~general~~ formula sheet

$\therefore u(x,t) = v(x,t) + u_p(x)$

Answer: $= -\sum_{k=1}^{\infty} \frac{10}{k\pi}(1-(-1)^k) \sin(k\pi x) e^{-k^2\pi^2 t}$

$+ \sum_{k=1}^{100} 2^{-k} \sin(k\pi x) e^{-k^2\pi^2 t}$

$+ 5.$

End of Exam