Math 267, Section 202 : HW 9

Due **Wednesday, March 27th**.

1. [Convolution of non-periodic signals]

   Recall for integers $n \in \mathbb{Z}$,

   $$u[n] = \begin{cases} 1 & \text{if } n \geq 0 \text{ ,} \\ 0 & \text{otherwise.} \end{cases} \qquad \delta[n] = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \qquad \delta_{n_0}[n] = \begin{cases} 1 & \text{if } n = n_0, \\ 0 & \text{otherwise.} \end{cases}$$

   Recall the class example $(u * u)[n] = (n+1)u[n]$.

   (a) Find $\overbrace{(\delta_2 * \delta_2 * \cdots * \delta_2)}^{100 \text{ times}}[n]$.

   (b) Let $f[n] = u[n-2]$. $g[n] = u[n+3]$.

   i. Find $(f * u)[n]$.

   ii. Find $(f * g)[n]$.

   (c) Let

   $$h[n] = \begin{cases} 1 & |n| \leq 3 \text{ ,} \\ 0 & \text{otherwise.} \end{cases}$$

   Find $(h * u)[n]$

   i. first, by computing the convolution sum directly;

   ii. second, by using the algebraic properties of the convolution and using $(u * u)[n] = (n+1)u[n]$.

Solution

(a): Recall $\delta_a[n] = \delta[n-a]$ for $a \in \mathbb{Z}$. $\delta_a * \delta_b = \delta_{a+b}$. By associativity, $\delta_a * \delta_b * \delta_c = \delta_{a+b+c}$, and so on. Thus, $\overbrace{(\delta_2 * \delta_2 * \cdots * \delta_2)}^{100 \text{ times}}[n] = \delta_{2\times 100}[n] = \underline{\delta_{200}[n]}$.

(b); Note that we can write $f[n] = (\delta_2 * u)[n]$ and $g[n] = (\delta_{-3} * u)[n]$. Thus,

$$(f * u)[n] = (\delta_2 * u * u)[n] = (u * u)[n-2] = (n-2+1)u[n-2] = \underline{(n-1)u[n-2]}$$

$$(f * g)[n] = (\delta_2 * u * \delta_{-3} * u)[n] = (\delta_2 * \delta_{-3} * u * u)[n] = (\delta_{-1} * u * u)[n] = (n+1+1)u[n+1] = \underline{(n+2)u}$$

(c):

(i):

$$\begin{aligned}(h*u)[n] &= \sum_{m=-\infty}^{\infty} h[m]u[n-m] = \sum_{m=-\infty}^{n} h[m]u[n-m] \qquad \text{(require } n-m\geq 0\text{)}\\ &= \sum_{m=-\infty}^{n} h[m]\\ &= \underline{\begin{cases} 0 & \text{for } n<-3,\\ n+4 & \text{for } -3\leq n\leq 3,\\ 7 & \text{for } n>3.\end{cases}}\end{aligned}$$

(ii) Note that $h[n] = u[n+3] - u[n-4] = (\delta_{-3} * u)[n] - (\delta_4 * u)[n]$. Therefore,

$$\begin{aligned}(h*u)[n] &= (\delta_{-3}*u*u)[n] - (\delta_4*u*u)[n] = (n+3+1)u[n+3] - (n-4+1)u[n-4]\\ &= \underline{(n+4)u[n+3] - (n-3)u[n-4]}\end{aligned}$$

Notice that

$$(n+4)u[n+3] - (n-3)u[n-4] = \begin{cases} 0 & \text{for } n<-3,\\ n+4 & \text{for } -3\leq n\leq 3,\\ (n+4)-(n-3)=7 & \text{for } n>3.\end{cases}$$

Thus, the answers in (i) and (ii) coincide.

2. [Discrete-time Fourier transform for non-periodic signals]

   (a) $x[n] = \delta_2[n] + \delta_{-2}[n]$

   (b) $y[n] = \left(\frac{1}{5}\right)^n u[n-1]$

   (c) $z[n] = \left(\frac{1}{5}\right)^{|n+1|}$

   Solution

   (a)

$$\widehat{x}(\omega) = \widehat{\delta}_2(\omega) + \widehat{\delta}_{-2}(\omega) = \underline{e^{-2i\omega} + e^{+2i\omega} = 2\cos(2\omega)}$$

   Here in the first equality, we used the linearity and in the second equality we used the time-shift property. (Of course, in this simple case, we can just apply the definition of Fourier transform and the Delta function.)

   (b): $y[n] = \left(\frac{1}{5}\right)^n u[n-1] = \left(\frac{1}{5}\right)\left(\frac{1}{5}\right)^{n-1} u[n-1]$. Thus, using time-shift (using $\mathcal{F}$ to denote the discrete-time Fourier transform),

$$\begin{aligned}\widehat{y}(\omega) &= \frac{1}{5}\mathcal{F}[\left(\frac{1}{5}\right)^{n-1}u[n-1]](\omega) = \frac{1}{5}e^{-i\omega}\mathcal{F}[\left(\frac{1}{5}\right)^n u[n]](\omega)\\ &= \underline{\frac{1}{5}e^{-i\omega}\frac{1}{1-\frac{1}{5}e^{-i\omega}}}\end{aligned}$$

(c) Notice that since $1 = u[n] + u[-n-1]$, one can write any $x[n]$ as $x[n] = x[n]u[n] + x[n]u[-n-1]$

$z[n] = \left(\frac{1}{5}\right)^{|n+1|} = \left(\frac{1}{5}\right)^{|n+1|}u[n] + \left(\frac{1}{5}\right)^{|n+1|}u[-n-1]$ which then can be written as

$$z[n] = \Big(\frac{1}{5}\Big)^{n+1}u[n] + \Big(\frac{1}{5}\Big)^{-(n+1)}u[-(n+1)]$$

Thus,

$$\widehat{z}(\omega) = \frac{1}{5}\mathcal{F}\Big[\Big(\frac{1}{5}\Big)^{n}u[n]\Big](\omega) + \mathcal{F}\Big[\Big(\frac{1}{5}\Big)^{-(n+1)}u[-(n+1)]\Big](\omega)$$

Note

$$\mathcal{F}\Big[\Big(\frac{1}{5}\Big)^{n}u[n]\Big](\omega) = \frac{1}{1-\frac{1}{5}e^{-i\omega}}$$

$$\begin{aligned}
\mathcal{F}\Big[\Big(\frac{1}{5}\Big)^{-(n+1)}u[-(n+1)]\Big](\omega) &= e^{+i\omega}\mathcal{F}\Big[\Big(\frac{1}{5}\Big)^{-n}u[-n]\Big](\omega) \qquad \text{(time-shift)}\\
&= e^{+i\omega}\mathcal{F}\Big[\Big(\frac{1}{5}\Big)^{n}u[n]\Big](-\omega) \qquad \text{(time-reversal)}\\
&= e^{i\omega}\frac{1}{1-\frac{1}{5}e^{i\omega}}
\end{aligned}$$

Therefore,

$$\widehat{z}(\omega) = \underline{\frac{1}{5}\times\frac{1}{1-\frac{1}{5}e^{-i\omega}} + e^{i\omega}\frac{1}{1-\frac{1}{5}e^{i\omega}}}$$

Remark: The time-reversal property is in the online notes page 12 in the table, and it can also be proved very easily:

$$\begin{aligned}
\mathcal{F}[x[-n]](\omega) &= \sum_{n=-\infty}^{\infty} x[-n]e^{-in\omega} = \sum_{m=-\infty}^{\infty} x[m]e^{-im(-\omega)} \qquad \text{change of index } m=-n\\
&= \mathcal{F}[x[n]](-\omega)
\end{aligned}$$

3. **[NOT TO HAND IN]** [Inverse discrete-time Fourier transform for non-periodic signals]

   Recall the discrete-time Fourier transforms of $\delta_{n_0}[n]$ and $a^n u[n]$ (for $|a| < 1$) are $e^{-i\omega n_0}$ and $\frac{1}{1-ae^{-i\omega}}$, respectively.

   Use these to find discrete-time signals $x[n]$, $y[n]$, $z[n]$, whose Fourier transforms are given below. (Here, each answer should be a signal defined on the set of integers: $n \in \mathbb{Z}$.)

   (a) $\widehat{x}(\omega) = \cos^2\omega + \cos\omega\sin\omega$. (Hint: Can we express this as combination of complex exponentials?)

(b)

$$\widehat{y}(\omega) = 1 + \frac{e^{i2\omega}}{1 + \frac{1}{3}e^{-i\omega}}$$

(Hint; you may want to use time-shift property: see the table in page 12 in the online note " Discrete-time Fourier series and Fourier Transforms".)

(c)

$$\widehat{z}(\omega) = \frac{1}{(1 + \frac{1}{2}e^{-i\omega})(1 + \frac{1}{3}e^{-i\omega})}.$$

(Hint: use partial fractions.)

Solution

(a): Note that

$$\begin{aligned}\cos^2\omega + \cos\omega\sin\omega &= \Big(\frac{e^{i\omega} + e^{-i\omega}}{2}\Big)^2 + \frac{e^{i\omega} + e^{-i\omega}}{2}\frac{e^{i\omega} - e^{-i\omega}}{2i} \\ &= \frac{1}{4}(e^{2i\omega} + 2 + e^{-2i\omega}) + \frac{1}{4i}(e^{2i\omega} - e^{-2i\omega})\end{aligned}$$

Thus, denoting the inverse discrete-time Fourier transform by $\mathcal{F}^{-1}$, we see

$$\begin{aligned}x[n] = \mathcal{F}^{-1}[\widehat{x}(\omega)][n] &= \frac{1}{4}(\mathcal{F}^{-1}[e^{2i\omega}][n] + \mathcal{F}^{-1}[2][n] + \mathcal{F}^{-1}[e^{-2i\omega}][n]) + \frac{1}{4i}(\mathcal{F}^{-1}[e^{2i\omega}][n] - \mathcal{F}^{-1}[e^{-2i\omega}][n]) \\ &= \frac{1}{4}(\delta[n+2] + 2\delta[n] + \delta[n-2]) + \frac{1}{4i}(\delta[n+2] - \delta[n-2]) \\ &= \begin{cases} \frac{1}{4} - \frac{1}{4i} & \text{for } n = 2\ , \\ \frac{1}{4} + \frac{1}{4i} & \text{for } n = -2\ , \\ \frac{1}{2} & \text{for } n = 0\ , \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

(b)

$$y[n] = \mathcal{F}^{-1}\Big[1\Big][n] + \mathcal{F}^{-1}\Big[\frac{e^{i2\omega}}{1 + \frac{1}{3}e^{-i\omega}}\Big][n]$$

Note that by the time-shift property, $4\frac{e^{i2\omega}}{1+\frac{1}{3}e^{-i\omega}} = \mathcal{F}[\big(-\frac{1}{3}\big)^{n+2}u[n+2]](\omega)$, therefore, $\mathcal{F}^{-1}\Big[\frac{e^{i2\omega}}{1+\frac{1}{3}e^{-i\omega}}\Big][n] = \big(-\frac{1}{3}\big)^{n+2}u[n+2]$. Thurs,

$$\underline{y[n] = \delta[n] + \big(-\frac{1}{3}\big)^{n+2}u[n+2]}$$

(c): Use partial fractions to see,

$$\begin{aligned}\widehat{z}(\omega) &= \frac{1}{(1+\frac{1}{2}e^{-i\omega})(1+\frac{1}{3}e^{-i\omega})} \\ &= \frac{A}{(1+\frac{1}{2}e^{-i\omega})} + \frac{B}{(1+\frac{1}{3}e^{-i\omega})}\end{aligned}$$

where

$$1 = A + \frac{A}{3}e^{-i\omega} + B + \frac{B}{2}e^{-i\omega}$$

Thus, $A + B = 1$ and $A/3 + B/2 = 0$, and $A = 3, B = -2$.
Thus,

$$\begin{aligned}z[n] &= 3\mathcal{F}^{-1}\Big[\frac{1}{1+\frac{1}{2}e^{-i\omega}}\Big][n] - 2\mathcal{F}^{-1}\Big[\frac{1}{1+\frac{1}{3}e^{-i\omega}}\Big][n] \\ &= 3\big(-\frac{1}{2}\big)^n u[n] + 2\big(-\frac{1}{3}\big)^n u[n] \\ &= \underline{\Big(3\big(-\frac{1}{2}\big)^n - 2\big(-\frac{1}{3}\big)^n\Big)u[n]} \\ &= \underline{\begin{cases}3\big(-\frac{1}{2}\big)^n - 2\big(-\frac{1}{3}\big)^n & \text{for } n \geq 0\ , \\ 0 & \text{otherwise.}\end{cases}}\end{aligned}$$

Second method: Use the convolution property: for $\widehat{x}(\omega) = \frac{1}{1+\frac{1}{2}e^{-i\omega}}$, $\quad \widehat{y}(\omega) = \frac{1}{1+\frac{1}{3}e^{-i\omega}}$, we see
$z[n] = (x * y)[n]$.
Notice that $x[n] = \big(-\frac{1}{2}\big)^n u[n]$, $\qquad y[n] = \big(-\frac{1}{3}\big)^n u[n]$.

Therefore,

$$
\begin{aligned}
z[n] &= \sum_{m=-\infty}^{\infty} x[m]y[n-m] \\
&= \sum_{m=-\infty}^{\infty} \left(-\frac{1}{2}\right)^m u[m] \left(-\frac{1}{3}\right)^{n-m} u[n-m] \\
&= \left(-\frac{1}{3}\right)^n \sum_{m=-\infty}^{\infty} \left(\frac{3}{2}\right)^m u[m]u[n-m] \qquad \text{(require } m \geq 0 \text{ and } n-m \geq 0 \text{)} \\
&= \begin{cases} \left(-\frac{1}{3}\right)^n \sum_{m=0}^{n} \left(\frac{3}{2}\right)^m & \text{for } n \geq 0 \text{ ,} \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} \left(-\frac{1}{3}\right)^n \left(\frac{1-\left(\frac{3}{2}\right)^{n+1}}{1-\frac{3}{2}}\right) & \text{for } n \geq 0 \text{ ,} \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} -2\left(-\frac{1}{3}\right)^n + 3\left(-\frac{1}{2}\right)^n & \text{for } n \geq 0 \text{ ,} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Check that the two methods gave the same answer.

4. Let $x[n] = \left(\frac{1}{3}\right)^n u[n]$ and $y[n] = \left(\frac{1}{5}\right)^n u[n]$

   (a) Find $(x * y)[n]$ by directly computing the convolution summation.
   Solution

$$
\begin{aligned}
(x * y)[n] &= \sum_{m=-\infty}^{\infty} x[m]y[n-m] \\
&= \sum_{m=-\infty}^{\infty} \left(\frac{1}{3}\right)^m \left(\frac{1}{5}\right)^{n-m} u[m]u[n-m] \\
&= \left(\frac{1}{5}\right)^n \sum_{m=-\infty}^{\infty} \left(\frac{1}{3}\right)^m \left(\frac{1}{5}\right)^{-m} u[m]u[n-m] \\
&= \left(\frac{1}{5}\right)^n \sum_{m=-\infty}^{\infty} \left(\frac{5}{3}\right)^m u[m]u[n-m] \\
&= \left(\frac{1}{5}\right)^n u[n] \sum_{m=0}^{n} \left(\frac{5}{3}\right)^m \\
&= \left(\frac{1}{5}\right)^n u[n] \frac{1-\left(\frac{5}{3}\right)^{n+1}}{1-\frac{5}{3}} \\
&= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{3}\right)^{n+1} - 1}{\frac{2}{3}} u[n]
\end{aligned}
$$

(b) Find $(x*y)[n]$ by applying DTFT.

Solution (Note $\frac{1}{3}, \frac{1}{5} < 1$, so we can apply the basic example for the DTFT.)

$$x[n] = \left(\frac{1}{3}\right)^n u[n] \rightarrow \text{ DTFT } \rightarrow \frac{1}{1-\frac{1}{3}e^{-i\omega}}$$
$$y[n] = \left(\frac{1}{5}\right)^n u[n] \rightarrow \text{ DTFT } \rightarrow \frac{1}{1-\frac{1}{5}e^{-i\omega}}$$

Now

$$(x*y)[n] = \rightarrow \text{ DTFT } \rightarrow \frac{1}{1-\frac{1}{3}e^{-i\omega}}\frac{1}{1-\frac{1}{5}e^{-i\omega}}$$

Here, partial fraction gives

$$\frac{1}{1-\frac{1}{3}e^{-i\omega}}\frac{1}{1-\frac{1}{5}e^{-i\omega}} = \frac{1}{2}\Big(\frac{5}{1-\frac{1}{3}e^{-i\omega}} - \frac{3}{1-\frac{1}{5}e^{-i\omega}}\Big)$$

Now, using basic examples, we see

$$\frac{1}{2}\Big(\frac{5}{1-\frac{1}{3}e^{-i\omega}} - \frac{3}{1-\frac{1}{5}e^{-i\omega}}\Big) \rightarrow \text{ inverse DTFT } \rightarrow \frac{1}{2}\Big(5\left(\frac{1}{3}\right)^n u[n] - 3\left(\frac{1}{5}\right)^n u[n]\Big)$$

Therefore,

$$\begin{aligned}(x*y)[n] &= \frac{1}{2}\Big(5\left(\frac{1}{3}\right)^n u[n] - 3\left(\frac{1}{5}\right)^n u[n]\Big) \\ &= \underline{\frac{1}{2}\Big(5\left(\frac{1}{3}\right)^n - 3\left(\frac{1}{5}\right)^n\Big)u[n]}\end{aligned}$$

(c) Check whether you get the same answer from (a) and (b).

Solution Let us start with the answer from (a)

$$\begin{aligned}&\left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{3}\right)^{n+1} - 1}{\frac{2}{3}} u[n] \\ &= \frac{\left(\frac{1}{5}\right)^n\left(\frac{5}{3}\right)^{n+1} - \left(\frac{1}{5}\right)^n}{\frac{2}{3}} u[n] \\ &= \frac{\frac{5}{3}\left(\frac{1}{3}\right)^n - \left(\frac{1}{5}\right)^n}{\frac{2}{3}} u[n] \\ &= \frac{1}{2}\Big(5\left(\frac{1}{3}\right)^n - 3\left(\frac{1}{5}\right)^n\Big)u[n]\end{aligned}$$

and in the last line get the same answer as in (b).

5. Use DTFT to find a discrete time signal $y[n]$ that satisfies

$$y[n] - \frac{1}{4}y[n-2] = \delta[n-2] \quad \text{for all integer } n.$$

(Hint: you may need to do partial fractions.)

Solution

Using the time-shift property and a basic example:

$$\begin{aligned}
\widehat{y}_{DT}(\omega) - \frac{1}{4}e^{-i\omega\,2}\widehat{y}_{DT}(\omega) &= e^{-i\omega\,2} \\
\widehat{y}_{DT}(\omega) &= e^{-i\omega\,2} \cdot \frac{1}{1-\frac{1}{4}e^{-i\omega\,2}} \\
\widehat{y}_{DT}(\omega) &= e^{-i\omega\,2}\left(\frac{1}{\left(1+\frac{1}{2}e^{-i\omega}\right)\left(1-\frac{1}{2}e^{-i\omega}\right)}\right) \\
\widehat{y}_{DT}(\omega) &= \frac{1}{2}e^{-i\omega\,2}\left(\frac{1}{1+\frac{1}{2}e^{-i\omega}} + \frac{1}{1-\frac{1}{2}e^{-i\omega}}\right)
\end{aligned}$$

Which can be seen as either a time-shift of the other basic example, or as the product of the two basic examples. Under the second interpretation:

$$\begin{aligned}
y[n] &= \frac{1}{2}\delta_2[n] * \left(\left(-\frac{1}{2}\right)^n u[n]\right) + \frac{1}{2}\delta_2[n] * \left(\left(\frac{1}{2}\right)^n u[n]\right) \\
&= \frac{1}{2}\left(-\frac{1}{2}\right)^{n-2} u[n-2] + \frac{1}{2}\left(\frac{1}{2}\right)^{n-2} u[n-2]
\end{aligned}$$

6. Consider an LTI system given by the following difference equation:

$$y[n] - 3y[n-1] = x[n] \qquad \text{for all integers } n.$$

(a) Find the impulse response function $h[n]$ satisfying $h[n] = 0$ for all $n < 0$. For this case, find $y[n]$ when $x[n] = u[n]$.

Solution

To compute $h[n]$: We have these equations:

$$\begin{cases} y[\bar{n}] - 3y[\bar{n}-1] = 0 & \text{for } \bar{n} < 0 \\ y[0] - 3y[-1] = 1 & \\ y[\bar{n}] - 3y[\bar{n}-1] = 0 & \text{for } \bar{n} > 0 \end{cases}$$

From the equations when $\bar{n} < 0$:

$$\begin{aligned}
y[-1] = 3y[-2] \quad &\rightarrow \quad y[-2] = \frac{1}{3}y[-1] \\
y[-2] = 3y[-3] \quad &\rightarrow \quad y[-3] = \frac{1}{3}y[-2] = \frac{1}{3^2}y[-1] \\
y[-3] = 3y[-4] \quad &\rightarrow \quad y[-4] = \frac{1}{3}y[-3] = \frac{1}{3^3}y[-2] \\
&\dots
\end{aligned}$$

We conclude that $y[n] = 3^{n+1}y[-1]$ for $n < 0$. To make these all zero, we must have $y[-1] = 0$.
From the middle equation:

$$y[0] - 3y[-1] = 1 \quad \rightarrow \quad y[0] = 1$$

From the equations when $\bar{n} > 0$:

$$\begin{aligned}
y[1] = 3y[0] \quad &\rightarrow \quad y[1] = \frac{1}{3}y[0] = \frac{1}{3} \\
y[2] = 3y[1] \quad &\rightarrow \quad y[2] = \frac{1}{3}y[1] = \frac{1}{3^2} \\
y[3] = 3y[2] \quad &\rightarrow \quad y[3] = \frac{1}{3}y[2] = \frac{1}{3^3} \\
\ldots
\end{aligned}$$

We conclude that $y[n] = 3^n$ for $n \geq 0$. All together:

$$h[n] = 3^n u[n]$$

To compute $y[n]$: To calculate the output $y[n]$ from the input $x[n] = u[n]$, use $y = h * x$.

$$\begin{aligned}
y[n] = (h * x)[n] &= \sum_{k=-\infty}^{\infty} 3^k u[k]\, u[n-k] \\
&= u[n] \sum_{k=0}^{n} 3^k \cdot 1 \\
&= u[n] \left(\frac{1 - 3^{n+1}}{1 - 3} \right)
\end{aligned}$$

(b) Find the impulse response function $h[n]$ satisfying $h[0] = 0$. For this case, find $y[n]$ when $x[n] = u[n]$.

Solution

To compute $h[n]$: We have these equations:

$$\begin{cases}
y[\bar{n}] - 3y[\bar{n} - 1] = 0 & \text{for } \bar{n} < 0 \\
y[0] - 3y[-1] = 1 & \\
y[\bar{n}] - 3y[\bar{n} - 1] = 0 & \text{for } \bar{n} > 0
\end{cases}$$

From the equations when $\bar{n} < 0$:

$$\begin{aligned}
y[-1] = 3y[-2] \quad &\rightarrow \quad y[-2] = \frac{1}{3}y[-1] \\
y[-2] = 3y[-3] \quad &\rightarrow \quad y[-3] = \frac{1}{3}y[-2] = \frac{1}{3^2}y[-1] \\
y[-3] = 3y[-4] \quad &\rightarrow \quad y[-4] = \frac{1}{3}y[-3] = \frac{1}{3^3}y[-2] \\
\ldots
\end{aligned}$$

We conclude that $y[n] = 3^{n+1} y[-1]$ for $n < 0$.
From the middle equation and the condition $y[0] = 0$:

$$y[0] - 3y[-1] = 1 \quad \rightarrow \quad y[-1] = -\frac{1}{3}$$

Thus, we determine,

$$y[n] = -3^{n+1}\tfrac{1}{3} = -3^n \text{ for } n < 0.$$

From the equations when $\bar{n} > 0$:

$$\begin{aligned} y[1] &= 3y[0] \quad \rightarrow \quad y[1] = \frac{1}{3}y[0] = 0 \\ y[2] &= 3y[1] \quad \rightarrow \quad y[2] = \frac{1}{3}y[1] = 0 \\ y[3] &= 3y[2] \quad \rightarrow \quad y[3] = \frac{1}{3}y[2] = 0 \\ \dots \end{aligned}$$

We conclude that $y[n] = 0$ for $n \geq 0$. All together:

$$h[n] = -3^n u[-n-1]$$

<u>To compute $y[n]$</u>: To calculate the output $y[n]$ from the input $x[n] = u[n]$, use $y = h * x$.

$$y[n] = (h * x)[n] = \sum_{k=-\infty}^{\infty} -3^k u[-k-1]\, u[n-k]$$

In the last sum, we see that we have nontrivial terms to add only when $-k - 1 \geq 0$ and $k \leq n$, that is, $k \leq -1$ and $k \leq n$.
Two cases:
<u>Case: $n \geq 0$</u>

$$\begin{aligned} y[n] = (h * x)[n] &= \sum_{k=-\infty}^{\infty} -3^k u[-k-1]\, u[n-k] \quad = \sum_{k=-\infty}^{-1} -3^k = -\sum_{m=1}^{m} 3^{-m} \\ &= -\frac{1}{3}\sum_{m=0}^{\infty} 3^{-m} \\ &= -\frac{1}{3}\frac{1}{1-\frac{1}{3}} = -\frac{1}{2} \end{aligned}$$

Case: $n \leq -1$

$$\begin{aligned}
y[n] = (h * x)[n] &= \sum_{k=-\infty}^{\infty} -3^k u[-k-1]\, u[n-k] \\
&= \sum_{k=-\infty}^{n} -3^k \\
&= -3^n \sum_{k=-\infty}^{n} 3^{k-n} \\
&= -3^n \sum_{m=-\infty}^{0} 3^m \quad \text{(change of index } m = k-n\text{)} \\
&= -3^n \sum_{l=0}^{\infty} 3^{-l} \quad \text{(change of index } l = -m\text{)} \\
&= -3^n \frac{1}{1-\frac{1}{3}} = -\frac{3^{n+1}}{2}
\end{aligned}$$

Combining the two cases, we see

$$y[n] = \underline{-\frac{1}{2}u[n] - \frac{3^{n+1}}{2}u[-n-1]}$$