Math 267, Section 202 : HW 8

Due **Wednesday, March 20th**.

In the following, we do not distinguish between "length= N discrete-time signals" and "N-periodic discrete-time signals".

1. **[NOT TO HAND-IN]** For the given discrete-time signal x, find $\widehat{x}$, i.e. its discrete Fourier transform (or in other words, the Fourier coefficient of discrete Fourier series).

   (a) $x = [0, 1, 0, 0]$.

   (b) $x = [1, 1, 1, 1]$.

   (c) $x = [1, -1, 1, -1]$.

   (d) $x = [1, 2, 3, 4]$.

   (e) $x = [1, \frac{1}{3}, \frac{1}{3^2}, \cdots, \frac{1}{3^{10}}, \frac{1}{3^{11}}]$.

2. Calculate the DFT (in other words, the Fourier coefficient $\widehat{x}[k]$ of Discrete Fourier Series) for

$$x[n] = 3^{-|n-10|}, \quad \text{for } n = 0, \ldots, 41$$

   Here $N = 42$.

3. [Discrete complex exponentials]

   (a) Compute

$$\sum_{n=0}^{9} \left[\left(e^{-i\frac{2\pi}{10}2n} + e^{i\frac{2\pi}{10}3n} \right) \left(e^{i\frac{2\pi}{10}2n} + e^{-i\frac{2\pi}{10}3n} + e^{i\frac{2\pi}{10}8n} \right) \right]$$

   (b) Compute

$$\sum_{n=0}^{9} \left[\left(e^{-i\frac{2\pi}{9}2n} + e^{i\frac{2\pi}{9}3n} \right) \left(e^{i\frac{2\pi}{9}2n} + e^{-i\frac{2\pi}{9}3n} + e^{i\frac{2\pi}{9}7n} \right) \right]$$

   (Hint: This (b) is tricker than (a).)

4. [Discrte Fourier transform for periodic signals]

   (a) Find the discrete Fourier transform (i.e. Discrete Fourier Series) of the following periodic signals with period N.

      i. $x[n] = \cos(2\pi n)$. $N = 4$

      ii. $y[n] = \cos(\pi n/3) + \sin(\pi n/2)$. $N = 12$

(Hint: Express sin and cos using complex exponentials and try to use 'orthogonality' to compute the summation.)

(b) Suppose $x[n]$ is a periodic discrete-time signal with period $= N$. Let $\widehat{x}[k]$ be its discrete Fourier transform. Assume $x[0] = N$ and $\sum_{k=0}^{N-1} |\widehat{x}[k]|^2 = N$. Find $x[n]$ and $\widehat{x}[k]$ for all $0 \leq n, k \leq N-1$. (Hint: Use Parseval's relation. This is similar to one of the class examples.)

(c) Let $a[n]$ be a periodic signal with period $N = 16$ with

$$a[n] = \begin{cases} 1 & 0 \leq n \leq 8, \\ 0 & 9 \leq n \leq 12, \\ 1 & 13 \leq n \leq 15. \end{cases}$$

Compute the discrete Fourier transform $\widehat{a}[k]$. (This is similar to one of the class examples.)

5. Calculate the DFT (in other words, the Fourier coefficient $\widehat{x}[k]$ of Discrete Fourier Series), and **fully simplify, if possible**:

(a) $x[n] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

(b) $x[n] = \begin{cases} 5^{-n} & \text{for } 4 \leq n \leq 8, \\ 0 & \text{for } 1 \leq n \leq 3. \end{cases}$ Here, $N = 8$.

6. **[NOT TO HAND-IN]** [Inverse discrete Fourier transform] Given $\widehat{x}$ in the following, find the original signal x by using the inverse discrete Fourier transform.

(a) $\widehat{x} = [0, 0, 3, 0]$.

(b) $\widehat{x} = [1, 1, 1, 1]$.

(c) $\widehat{x} = [1, \frac{1}{4}, \frac{1}{4^2}, \cdots, \frac{1}{4^{10}}, \frac{1}{4^{11}}]$

7. Find $x[n]$ given that:

(a) $\widehat{x}[k] = \sin\left(\frac{3\pi}{4}k\right) - \cos\left(\frac{5\pi}{4}k\right)$. Here, you first have to find the fundamental period N (i.e. the smallest period).

(b) $\widehat{x}[k] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

8. Let $x = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$ and $y = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{4} \end{bmatrix}$. Calculate the periodic convolution $x * y$.

9. **[NOT TO HAND-IN]** [Periodic convolution]

Consider the folloing signals with period $N = 4$:

$$a = [1, 0, 1, -1], \qquad b = [2, i, 1 + i, 3]$$

(e.g. $a[0] = 1, a[3] = -1, b[2] = 1 + i$, etc.)

(a) Calculate the periodic convolution $a * b$ **by directly calculating the convolution sum**.

(b) Calculate the Fourier coefficients $\widehat{a}[k]$ and $\widehat{b}[k]$. Use this to compute the Fourier coefficients $\widehat{a * b}[k]$ for $a * b$ by using the convolution property of the Fourier transform.

(c) Find a signal $x[n]$ of period $N = 4$, such that $(a * x)[n] = b[n]$.

(Hint: you may want to use the convolution property of the Fourier transform/inversion. Remember how we handle the circuit problem. This is similar.)

10. **[NOT TO HAND-IN]** Consider two discrete signals, both length $N = 3$:

$x[n] = [x[0], x[1], x[2]]$, and, $y[n] = [y[0], y[1], y[2]]$.

(a) Write out the definition of $(x * y)[0]$.

Think of $x[n]$ as a column vector $\vec{x}$.

Recognize that the formula for $(x * y)[0]$ is a dot-product of some row vector $\vec{a}$ with $\vec{x}$.

What are the components of vector $\vec{a}$?

(b) Repeat part(a) for $(x * y)[1]$ and $(x * y)[2]$.

Put all three row vectors into a matrix Y, so that,

$$(x * y) = Y\vec{x}.$$

(c) Use your answer to part(b) to compute $(x * y)$, where $x[n]$ and $y[n]$ are the following signals with length $N = 3$,

$$x = [2, -1, 1], \quad y = [-3, i, 1]$$