Math 267, Section 202 : HW 8

Due **Wednesday, March 20th**.

In the following, we do not distinguish between "length= N discrete-time signals" and "N-periodic discrete-time signals".

1. **[NOT TO HAND-IN]** For the given discrete-time signal x, find $\widehat{x}$, i.e. its discrete Fourier transform (or in other words, the Fourier coefficient of discrete Fourier series).

   (a) $x = [0, 1, 0, 0]$.
   Solution Note $e^{-i2\pi/4k} = e^{-ik\pi/2} = 1, -i, -1, i$ for $k = 0, 1, 2, 3$, respectively. Therefore,

   $$\widehat{x}[0] = \frac{1}{4}(0+1+0+0) = \frac{1}{4} \qquad \widehat{x}[1] = \frac{1}{4}(0+(-i)+0+0) = -\frac{i}{4}$$
   $$\widehat{x}[1] = \frac{1}{4}(0+(-i)^2+0+0) = -\frac{1}{4} \qquad \widehat{x}[3] = \frac{1}{4}(0+(-i)^3+0+0) = \frac{i}{4}.$$

   Thus,

   $$\underline{\widehat{x} = [\frac{1}{4}, -\frac{i}{4}, -\frac{1}{4}, \frac{i}{4}]}$$

   Note that this is nothing but $\widehat{x}[k] = e^{ik\pi/2}/4$ for $k = 0, 1, 2, 3$.

   (b) $x = [1, 1, 1, 1]$.
   Solution Note $e^{-i\pi/4k} = e^{-ik\pi/2} = 1, -i, -1, i$ for $k = 0, 1, 2, 3$, respectively. Therefore,

   $$\widehat{x}[0] = \frac{1}{4}(1+1+1+1) = 1 \qquad \widehat{x}[1] = \frac{1}{4}(1+(-i)-1+i) = 0$$
   $$\widehat{x}[1] = \frac{1}{4}(1+(-i)^2+(-1)^2+(i)^2) = 0 \qquad \widehat{x}[3] = \frac{1}{4}(1+(-i)^3+(-1)^3+i^3) = 0.$$

   (You can also use 'orthogonality' to see the above immediately.)
   Thus,

   $$\underline{\widehat{x} = [1, 0, 0, 0]}$$

   Remark Let us consider more general case: $x = \overbrace{[1, 1, \cdots, 1]}^{N \text{ entries}}$. By definition,

   $$\widehat{x}[k] = \frac{1}{N}\left[1 + e^{-2\pi i \frac{k}{N}} + e^{-2\pi i \frac{2k}{N}} + \cdots + e^{-2\pi i \frac{(N-1)k}{N}}\right]$$

Clearly $\widehat{x}[0] = \frac{1}{N}[N] = 1$. For $1 \le k \le N-1$, let $w = w_N = e^{\frac{2\pi i}{N}}$ to get

$$\widehat{x}[k] = \frac{1}{N}\left[1+w^{-k}+w^{-2k}+\cdots+w^{-(N-1)k}\right] = \frac{1}{N}\frac{1-w^{-Nk}}{1-w^{-k}} = 0 \qquad \text{since } w^{-kN} = e^{-2\pi k i} = 1$$

Hence $\widehat{x} = [1, 0, \cdots, 0]$ (N entries).

(c) $x = [1, -1, 1, -1]$.
Solution Note $e^{-i2\pi/4k} = e^{-ik\pi/2} = 1, -i, -1, i$ for $k = 0, 1, 2, 3$, respectively. Therefore,

$$\widehat{x}[0] = \frac{1}{4}(1-1+1-1) = 0 \qquad \widehat{x}[1] = \frac{1}{4}(1-(-i)-1-i) = 0$$
$$\widehat{x}[1] = \frac{1}{4}(1-(-i)^2+(-1)^2-i^2) = 1 \qquad \widehat{x}[3] = \frac{1}{4}(1-(-i)^3+(-1)^3-i^3) = 0.$$

(You can also use 'orthogonality' to see the above immediately, since $x[n] = (-1)^n$ for $n = 0, 1, 2, 3$.)
Thus,

$$\underline{\widehat{x} = [0, 0, 1, 0]}$$

(d) $x = [1, 2, 3, 4]$.
Solution
Note $e^{i2\pi/4k} = e^{ik\pi/2} = 1, i, -1, -i$ for $k = 0, 1, 2, 3$, respectively. Therefore,

$$\widehat{x}[0] = \frac{1}{4}(1-2+3-4) = -1/2 \qquad \widehat{x}[1] = \frac{1}{4}(1+2(-i)-3+4i) = \frac{-1+i}{2}$$
$$\widehat{x}[1] = \frac{1}{4}(1+2(-i)^2+3(-1)^2+4i^2) = -\frac{1}{2} \qquad \widehat{x}[3] = \frac{1}{4}(1+2(-i)^3+3(-1)^3+4i^3) = \frac{-1-i}{2}.$$

Thus,

$$\underline{\widehat{x} = [-\frac{1}{2}, \frac{-1+i}{2}, -\frac{1}{2}, \frac{-1-i}{2}]}$$

(e) $x = [1, \frac{1}{3}, \frac{1}{3^2}, \cdots, \frac{1}{3^{10}}, \frac{1}{3^{11}}]$.
Solution
Let us consider more general case: $x = [1, r, r^2, \cdots, r^{N-1}]$. (In the problem, $N = 12$, $r = 1/3$.)
Let $w = w_N = e^{\frac{2\pi i}{N}}$ Then,

$$\widehat{x}[0] = \frac{1}{N}\left[1+r+r^2+\cdots+r^{(N-1)}\right] = \frac{1}{N}\frac{1-r^N}{1-r}$$

(note that we could use the geometric sum for the case $r \neq 1$) and

$$\widehat{x}[k] = \frac{1}{N}\left[1 + rw^{-k} + \left(rw^{-k}\right)^2 + \cdots + \left(rw^{-k}\right)^{(N-1)}\right] = \frac{1-(rw^{-k})^N}{N(1-rw^{-k})} = \frac{1-r^N}{N(1-rw^{-k})}$$

for $1 \leq k \leq N-1$. (Here, we used that $w^N = 1$.)
Back to our specific case, we see that

$$\underline{\widehat{x}[k] = \frac{1-(\frac{1}{3})^{12}}{12(1-\frac{1}{3}e^{-i\pi k/6})}}$$

for $k = 0, 1, 2, \cdots, 11$.

2. Calculate the DFT (in other words, the Fourier coefficient $\widehat{x}[k]$ of Discrete Fourier Series) for

$$x[n] = 3^{-|n-10|}, \quad \text{for } n = 0, \ldots, 41$$

Here $N = 42$.

Solution

Immediately, split the sum into $n = 0, \ldots, 9$ and $n = 10, \ldots, 41$, in order to remove the absolute value:

$$\begin{aligned}
\widehat{x}[k] &= \sum_{n=0}^{41} x[n] e^{-i\frac{2\pi}{42}k\,n} \\
&= \sum_{n=0}^{9} 3^{(n-10)} e^{-i\frac{2\pi}{42}k\,n} + \sum_{n=10}^{41} 3^{-(n-10)} e^{-i\frac{2\pi}{42}k\,n} \\
&= 3^{-10} \sum_{n=0}^{9} \left(3\, e^{-i\frac{2\pi}{42}k}\right)^n + \sum_{\ell=0}^{31} 3^{-\ell} e^{-i\frac{2\pi}{42}k\,(\ell+10)} \\
&= 3^{-10} \sum_{n=0}^{9} \left(3\, e^{-i\frac{2\pi}{42}k}\right)^n + \left(e^{-i\frac{2\pi}{42}k}\right)^{10} \sum_{\ell=0}^{31} \left(3^{-1} e^{-i\frac{2\pi}{42}k}\right)^{\ell} \\
&= 3^{-10} \left(\frac{1-\left(3\, e^{-i\frac{2\pi}{42}k}\right)^{10}}{1-\left(3\, e^{-i\frac{2\pi}{42}k}\right)}\right) + \left(e^{-i\frac{2\pi}{42}k}\right)^{10} \left(\frac{1-\left(3^{-1} e^{-i\frac{2\pi}{42}k}\right)^{32}}{1-\left(3^{-1} e^{-i\frac{2\pi}{42}k}\right)}\right)
\end{aligned}$$

You are not required to simplify.

3. [Discrete complex exponentials]

(a) Compute

$$\sum_{n=0}^{9} \left[\left(e^{-i\frac{2\pi}{10}2n} + e^{i\frac{2\pi}{10}3n}\right)\left(e^{i\frac{2\pi}{10}2n} + e^{-i\frac{2\pi}{10}3n} + e^{i\frac{2\pi}{10}8n}\right)\right]$$

Solution Note that in the following sum $\sum_{n=0}^{9}$ there are $N = 10$ terms. This matches well with $e^{i\frac{2\pi}{10}kn}$ to apply orthogonality of discrete complex exponentials.

$$\sum_{n=0}^{9}\left[\left(e^{-i\frac{2\pi}{10}2n} + e^{i\frac{2\pi}{10}3n}\right)\left(e^{i\frac{2\pi}{10}2n} + e^{-i\frac{2\pi}{10}3n} + e^{i\frac{2\pi}{10}8n}\right)\right]$$
$$= \sum_{n=0}^{9} e^{-i\frac{2\pi}{10}2n}e^{i\frac{2\pi}{10}2n} + \sum_{n=0}^{9} e^{-i\frac{2\pi}{10}2n}e^{-i\frac{2\pi}{10}3n} + \sum_{n=0}^{9} e^{-i\frac{2\pi}{10}2n}e^{i\frac{2\pi}{10}8n}$$
$$+ \sum_{n=0}^{9} e^{i\frac{2\pi}{10}3n}e^{i\frac{2\pi}{10}2n} + \sum_{n=0}^{9} e^{i\frac{2\pi}{10}3n}e^{-i\frac{2\pi}{10}3n} + \sum_{n=0}^{9} e^{i\frac{2\pi}{10}3n}e^{i\frac{2\pi}{10}8n}$$
$$= 10 + 0 + 0 + 0 + 10 + 0$$

In the last line we have used the orthogonality of discrete complex exponentials. (For the last term, notice that $3 + 8 = 11$ is not an integer multiple of 10.) So, the answer is

$$\underline{20}$$

(b) Compute

$$\sum_{n=0}^{9}\left[\left(e^{-i\frac{2\pi}{9}2n} + e^{i\frac{2\pi}{9}3n}\right)\left(e^{i\frac{2\pi}{9}2n} + e^{-i\frac{2\pi}{9}3n} + e^{i\frac{2\pi}{9}7n}\right)\right]$$

(Hint: This (b) is tricker than (a).)

Solution Notice that the sum $\sum_{n=0}^{9}$ has 10 terms, but, we have complex exponentials $e^{-i\frac{2\pi}{9}kn}$ with $N = 9$. Because of this we decompose the sum as

$$\sum_{n=0}^{9}\left[\left(e^{-i\frac{2\pi}{9}2n} + e^{i\frac{2\pi}{9}3n}\right)\left(e^{i\frac{2\pi}{9}2n} + e^{-i\frac{2\pi}{9}3n} + e^{i\frac{2\pi}{9}7n}\right)\right]$$
$$= \sum_{n=0}^{8}\left[\left(e^{-i\frac{2\pi}{9}2n} + e^{i\frac{2\pi}{9}3n}\right)\left(e^{i\frac{2\pi}{9}2n} + e^{-i\frac{2\pi}{9}3n} + e^{i\frac{2\pi}{9}7n}\right)\right]$$
$$+ \left(e^{-i\frac{2\pi}{9}2\times 9} + e^{i\frac{2\pi}{9}3\times 9}\right)\left(e^{i\frac{2\pi}{9}2\times 9} + e^{-i\frac{2\pi}{9}3\times 9} + e^{i\frac{2\pi}{9}7\times 9}\right)$$

For the summation $\sum_{n=0}^{8}$ part, we use the orthogonality of discrete complex exponentials, and see

$$\sum_{n=0}^{8}\left[\left(e^{-i\frac{2\pi}{9}2n}+e^{i\frac{2\pi}{9}3n}\right)\left(e^{i\frac{2\pi}{9}2n}+e^{-i\frac{2\pi}{9}3n}+e^{i\frac{2\pi}{9}7n}\right)\right]$$
$$=\sum_{n=0}^{8}e^{-i\frac{2\pi}{9}2n}e^{i\frac{2\pi}{9}2n}+\sum_{n=0}^{8}e^{-i\frac{2\pi}{9}2n}e^{-i\frac{2\pi}{9}3n}+\sum_{n=0}^{8}e^{-i\frac{2\pi}{9}2n}e^{i\frac{2\pi}{9}7n}$$
$$+\sum_{n=0}^{8}e^{i\frac{2\pi}{9}3n}e^{i\frac{2\pi}{9}2n}+\sum_{n=0}^{8}e^{i\frac{2\pi}{9}3n}e^{-i\frac{2\pi}{9}3n}+\sum_{n=0}^{8}e^{i\frac{2\pi}{9}3n}e^{i\frac{2\pi}{9}7n}$$
$$=9+0+0+0+9+0=18$$

For the remaining part,

$$\left(e^{-i\frac{2\pi}{9}2\times 9}+e^{i\frac{2\pi}{9}3\times 9}\right)\left(e^{i\frac{2\pi}{9}2\times 9}+e^{-i\frac{2\pi}{9}3\times 9}+e^{i\frac{2\pi}{9}7\times 9}\right)$$
$$=(1+1)(1+1+1)\qquad(\text{since } e^{i2\pi k}=1 \text{ for integer } k).$$
$$=6.$$

Therefore the final answer is $\underline{18+6=24}$.

4. [Discrte Fourier transform for periodic signals]

(a) Find the discrete Fourier transform (i.e. Discrete Fourier Series) of the following periodic signals with period N.

i. $x[n]=\cos(2\pi n)$. $N=4$

ii. $y[n]=\cos(\pi n/3)+\sin(\pi n/2)$. $N=12$

(Hint: Express sin and cos using complex exponentials and try to use 'orthogonality' to compute the summation.)

Solution (a): (i) Note $\cos(2\pi n)=1$. Thus, $x=[1,1,1,1]$. And

$$\begin{aligned}\widehat{x}[k]&=\frac{1}{4}\sum_{n=0}^{3}x[n]e^{-i2\pi kn/4}\\&=\frac{1}{4}\sum_{n=0}^{3}e^{-i2\pi kn/4}\\&=\begin{cases}1 & \text{for } k=0\ ,\\ 0 & \text{for } k=1,2,3.\end{cases}\end{aligned}$$

Thus, $\widehat{x}=[1,0,0,0]$.

(ii) $\cos(\pi n/3)+\sin(\pi n/2)=\frac{1}{2}(e^{i\pi n/3}+e^{-i\pi n/3})+\frac{1}{2i}(e^{i\pi n/2}-e^{-i\pi n/2})$.

Thus,

$$\begin{aligned}
\widehat{y}[k] &= \frac{1}{12}\sum_{n=0}^{11}\left[\frac{1}{2}(e^{i\pi n/3}+e^{-i\pi n/3})+\frac{1}{2i}(e^{i\pi n/2}-e^{-i\pi n/2})\right]e^{-i2\pi kn/12} \\
&= \frac{1}{24}\sum_{n=0}^{11}(e^{i\pi n/3}+e^{-i\pi n/3})e^{-i2\pi kn/12} \; + \frac{1}{24i}\sum_{n=0}^{11}(e^{i\pi n/2}-e^{-i\pi n/2})e^{-i2\pi kn/12} \\
&= \frac{1}{24}\sum_{n=0}^{11}e^{i\pi n/3}e^{-i2\pi kn/12}+\frac{1}{24}\sum_{n=0}^{11}e^{-i\pi n/3}e^{-i2\pi kn/12} \\
&\quad + \frac{1}{24i}\sum_{n=0}^{11}e^{i\pi n/2}e^{-i2\pi kn/12}-\frac{1}{24i}\sum_{n=0}^{11}e^{-i\pi n/2}e^{-i2\pi kn/12} \\
&= \frac{1}{24}\sum_{n=0}^{11}e^{i2\pi 2n/12}e^{-i2\pi kn/12}+\frac{1}{24}\sum_{n=0}^{11}e^{-i2\pi 2n/12}e^{-i2\pi kn/12} \\
&\quad + \frac{1}{24i}\sum_{n=0}^{11}e^{i2\pi 3n/12}e^{-i2\pi kn/12}-\frac{1}{24i}\sum_{n=0}^{11}e^{-i2\pi 3n/12}e^{-i2\pi kn/12} \\
&= \frac{1}{24}\sum_{n=0}^{11}e^{i2\pi(2-k)n/12}+\frac{1}{24}\sum_{n=0}^{11}e^{i2\pi(-2-k)n/12} \\
&\quad + \frac{1}{24i}\sum_{n=0}^{11}e^{i2\pi(3-k)n/12}-\frac{1}{24i}\sum_{n=0}^{11}e^{i2\pi(-3-k)n/12} \\
&= \begin{cases} \frac{12}{24}+0+0-0 & k=2\ , \\ 0+\frac{12}{24}+0-0 & k=10\ , \\ 0+0+\frac{12}{24i}-0 & k=3, \\ 0+0+0-\frac{12}{24i} & k=9, \\ 0 & k=0,\cdots,11 \text{ but } k\neq 2,3,9,10. \end{cases} \qquad \text{(using orthogonality)}
\end{aligned}$$

Therefore,

$$\widehat{y}[k] = \begin{cases} \frac{1}{2} & k=2\ , \\ \frac{1}{2} & k=10, \\ \frac{1}{2i} & k=3, \\ -\frac{1}{2i} & k=9, \\ 0 & k=0,\cdots,11 \text{ but } k\neq 2,3,9,10. \end{cases}$$

In other words,

$$\widehat{y} = [0,0,\frac{1}{2},\frac{1}{2i},0,0,0,0,0,-\frac{1}{2i},\frac{1}{2},0]$$

(b) Suppose $x[n]$ is a periodic discrete-time signal with period $= N$. Let $\widehat{x}[k]$ be its discrete Fourier transform. Assume $x[0] = N$ and $\sum_{k=0}^{N-1} |\widehat{x}[k]|^2 = N$. Find $x[n]$ and $\widehat{x}[k]$ for all $0 \leq n, k \leq N-1$. (Hint: Use Parseval's relation. This is similar to one of the class examples.)

Solution

(b): Recall the Parseval's relation:

$$\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |\widehat{x}[k]|^2$$

Notice that all the entries in the summation are all nonnegative. By the given condition, the right hand side is N. Thus, we see $\sum_{n=0}^{N-1} |x[n]|^2 = N^2$. Now, since $x[0] = N$, the other entries in the last sum should all vanish, i.e. $x[1] = x[2] = \cdots = x[N-1] = 0$. Thus, $x = [1, 0, 0, \cdots, 0]$. Now, $\widehat{x}[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-2\pi ikn/N} = 1 + 0 + 0 + \cdots + 0 = 1$. Thus, $\underline{\widehat{x} = [1, 1, 1, \cdots, 1].}$

(c) Let $a[n]$ be a periodic signal with period $N = 16$ with

$$a[n] = \begin{cases} 1 & 0 \leq n \leq 8, \\ 0 & 9 \leq n \leq 12, \\ 1 & 13 \leq n \leq 15. \end{cases}$$

Compute the discrete Fourier transform $\widehat{a}[k]$. (This is similar to one of the class examples.)

Solution

(c) By time-shift we see that $a[n] = b[n+3]$ where $b[n]$ is the 16-periodic signal with

$$b[n] = \begin{cases} 1 & 0 \leq n \leq 11, \\ 0 & 12 \leq n \leq 15, \end{cases}$$

Therefore, $\widehat{a}[k] = e^{i\frac{2\pi}{16}\times 3k}\widehat{b}[k]$ and

$$\begin{aligned}
\widehat{b}[k] &= \frac{1}{16}\sum_{n=0}^{15} b[n]e^{-i2\pi kn/16} \\
&= \frac{1}{16}\sum_{n=0}^{11} e^{-i2\pi kn/16} \\
&= \frac{1}{16}\sum_{n=0}^{11} [e^{-i2\pi k/16}]^n \\
&= \frac{1}{16} \times \frac{1 - e^{-i\frac{2\pi k}{16}\times 12}}{1 - e^{-i2\pi k/16}}
\end{aligned}$$

Therefore,

$$\begin{aligned}\widehat{a}[k] &= e^{i\frac{2\pi}{16}\times 3k}\frac{1}{16}\times\frac{1-e^{-i\frac{2\pi k}{16}\times 12}}{1-e^{-i2\pi k/16}}\\ &= \underline{e^{i\frac{3\pi}{8}\times 3k}\frac{1}{16}\times\frac{1-e^{-i\frac{3\pi k}{2}}}{1-e^{-i\pi k/8}}}\end{aligned}$$

5. Calculate the DFT (in other words, the Fourier coefficient $\widehat{x}[k]$ of Discrete Fourier Series), and **fully simplify, if possible**:

   (a) $x[n] = \begin{bmatrix}1 & 0 & 1 & 0 & 0 & 0 & 1 & 0\end{bmatrix}$

   (b) $x[n] = \begin{cases} 5^{-n} & \text{for } 4 \le n \le 8, \\ 0 & \text{for } 1 \le n \le 3. \end{cases}$ Here, $N = 8$.

   Solution

   (a) Here, $N = 8$. Write the formula, then expand the sum:

$$\begin{aligned}\widehat{x}[k] &= \frac{1}{8}\sum_{n=0}^{7} x[n]e^{-i\frac{2\pi}{8}k\,n}\\ &= \frac{1}{8}\left(e^{-i\frac{2\pi}{8}k\,0}+e^{-i\frac{2\pi}{8}k\,2}+e^{-i\frac{2\pi}{8}k\,6}\right)\\ &= \frac{1}{8}\left(e^{-i\frac{2\pi}{8}k\,0}+e^{-i\frac{2\pi}{8}k\,2}+e^{-i\frac{2\pi}{8}k\,(-2)}\right)\\ &= \frac{1}{8}\left(1+e^{i\frac{\pi}{2}k}+e^{-i\frac{\pi}{2}k}\right)\\ &= \frac{1}{8}+\frac{1}{4}\cos(\frac{\pi}{2}k)\end{aligned}$$

   (b) Write the formula, plug in $x[n]$, then reorganize:

$$\begin{aligned}\widehat{x}[k] &= \frac{1}{8}\sum_{n=0}^{7} x[n]e^{-i\frac{2\pi}{8}k\,n}\\ &= \frac{1}{8}\sum_{n=1}^{8} x[n]e^{-i\frac{2\pi}{8}k\,n}\\ &= \frac{1}{8}\sum_{n=4}^{8} 5^{-n}e^{-i\frac{2\pi}{8}k\,n}\\ &= \frac{1}{8}\sum_{n=4}^{8}\left(\frac{e^{-i\frac{\pi}{4}k}}{5}\right)^{n}\end{aligned}$$

To get a proper geometric sum, substitute $\ell = n - 4$.

$$\begin{aligned}\widehat{x}[k] &= \frac{1}{8}\sum_{\ell=0}^{4}\left(\frac{e^{-i\frac{\pi}{4}k}}{5}\right)^{\ell+4} \\ &= \frac{1}{8}\left(\frac{e^{-i\frac{\pi}{4}k}}{5}\right)^{4}\sum_{\ell=0}^{4}\left(\frac{e^{-i\frac{\pi}{4}k}}{5}\right)^{\ell} \\ &= \frac{1}{8}\left(\frac{e^{-i\frac{\pi}{4}k}}{5}\right)^{4}\frac{1-\left(\frac{e^{-i\frac{\pi}{4}k}}{5}\right)^{5}}{1-\left(\frac{e^{-i\frac{\pi}{4}k}}{5}\right)}\end{aligned}$$

There's really no way to simplify further.

6. **[NOT TO HAND-IN]** [Inverse discrete Fourier transform] Given $\widehat{x}$ in the following, find the original signal x by using the inverse discrete Fourier transform.

   (a) $\widehat{x} = [0, 0, 3, 0]$.
   Solution Note $e^{i\pi/4k} = e^{ik\pi/2} = 1, i, -1, -i$ for $k = 0, 1, 2, 3$, respectively. Therefore,

   $$x[0] = 0 + 0 + 3 + 0 = 3 \qquad x[1] = 0 + 0 - 3 - 0 = -3$$
   $$x[1] = 0 + 0 + 3(-1)^2 + 0 = 3 \qquad x[3] = 0 + 0 + 3(-1)^3 + 0 = -3.$$

   Thus,

   $$\underline{x = [3, -3, 3, -3]}$$

   (b) $\widehat{x} = [1, 1, 1, 1]$.
   Solution
   Note $e^{i\pi/4k} = e^{ik\pi/2} = 1, i, -1, -i$ for $k = 0, 1, 2, 3$, respectively. Therefore,

   $$x[0] = 1 + 1 + 1 + 1 = 4 \qquad x[1] = 1 + i - 1 - i = 0$$
   $$x[1] = 1 + i^2 + (-1)^2 + (-i)^2 = 0 \qquad x[3] = 1 + i^3 + (-1)^3 + (-i)^3) = 0.$$

   (You can also use 'orthogonality' to see the above immediately.)
   Thus,

   $$\underline{x = [4, 0, 0, 0]}$$

   Remark Let us consider more general case: $\widehat{x} = \overbrace{[1, 1, \cdots, 1]}^{N \text{ entries}}$. By definition,

   $$x[n] = 1 + e^{2\pi i\frac{n}{N}} + e^{2\pi i\frac{2n}{N}} + \cdots + e^{2\pi i\frac{(N-1)n}{N}}$$

Clearly $x[0] = N$. For $1 \le n \le N-1$, let $w = w_N = e^{\frac{2\pi i}{N}}$ to get

$$x[n] = 1 + w^n + w^{2n} + \cdots + w^{(N-1)n} = \frac{1 - w^{Nn}}{1 - w^n} = 0 \qquad \text{since } w^{nN} = e^{2\pi n i} = 1$$

Hence $x = [N, 0, \cdots, 0]$, (N entries).

(c) $\widehat{x} = [1, \frac{1}{4}, \frac{1}{4^2}, \cdots, \frac{1}{4^{10}}, \frac{1}{4^{11}}]$

Solution

Let us consider more general case: $\widehat{x} = [1, r, r^2, \cdots, r^{N-1}]$. (In the problem, $N = 12$, $r = 1/4$.)

Let $w = w_N = e^{\frac{2\pi i}{N}}$. Then,

$$x[0] = 1 + r + r^2 + \cdots + r^{(N-1)} = \frac{1 - r^N}{1 - r}$$

(note that we could use the geometric sum for the case $r \neq 1$) and

$$x[n] = 1 + rw^n + \left(rw^n\right)^2 + \cdots + \left(rw^n\right)^{(N-1)}\Big] = \frac{1 - (rw^n)^N}{(1 - rw^n)} = \frac{1 - r^N}{1 - rw^n}$$

for $1 \le n \le N-1$. (Here, we used that $w^N = 1$.)

Back to our specific case, we see that

$$\underline{x[n] = \frac{1 - (\frac{1}{4})^{12}}{1 - \frac{1}{4} e^{i\pi n/6}}}$$

for $k = 0, 1, 2, \cdots, 11$.

7. Find $x[n]$ given that:

(a) $\widehat{x}[k] = \sin\left(\frac{3\pi}{4} k\right) - \cos\left(\frac{5\pi}{4} k\right)$. Here, you first have to find the fundamental period N (i.e. the smallest period).

(b) $\widehat{x}[k] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

Solution

(a) First: note $\sin(\frac{3\pi}{4} k)$ has period 8, as does $\cos(\frac{5\pi}{4} k)$, so $N = 8$.

We know the basic example:

$$\delta_c[n] \quad \underset{\text{DFT}}{\longrightarrow} \quad \frac{1}{N} e^{-i \frac{2\pi}{N} k\, c},$$

which means that if we write $\widehat{x}[k]$ in terms of complex exponentials we can read off the answer. (... or you can also answer this question using geometric sums.)

$$\begin{aligned} 8\,\widehat{x}[k] &= \frac{1}{2i} e^{i \frac{3\pi}{4} k} && - \frac{1}{2i} e^{-i \frac{3\pi}{4} k} && - \frac{1}{2} e^{i \frac{5\pi}{4} k} && - \frac{1}{2} e^{-i \frac{5\pi}{4} k} \\ &= \frac{1}{2i} e^{-i \frac{2\pi}{8} k(-3)} && - \frac{1}{2i} e^{-i \frac{2\pi}{8} k(+3)} && - \frac{1}{2} e^{-i \frac{2\pi}{8} k(-5)} && - \frac{1}{2} e^{-i \frac{2\pi}{8} k(+5)} \end{aligned}$$

We conclude: $x[n] = \frac{1}{8} \left(\frac{1}{2i} \delta_{-3}[n] - \frac{1}{2i} \delta_3[n] - \frac{1}{2} \delta_{-5}[n] - \frac{1}{2} \delta_5[n] \right)$.

(b) The inversion formula is easiest:

$$\begin{aligned} x[n] &= \sum_{k=0}^{N-1} \widehat{x}[k] e^{+i\frac{2\pi}{N}k\,n} \\ &= x[1]\, e^{+i\frac{2\pi}{N}(1)\,n} + x[-1]\, e^{+i\frac{2\pi}{N}(-1)\,n} \\ &= e^{+i\frac{2\pi}{8}n} - e^{-i\frac{2\pi}{8}n} = 2i\,\sin(\frac{\pi}{4}n) \end{aligned}$$

8. Let $x = \begin{bmatrix}0 & 1 & 1 & 0 & 0 & 0 & -1 & 0\end{bmatrix}$ and $y = \begin{bmatrix}\frac{1}{2} & \frac{1}{4} & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{4}\end{bmatrix}$. Calculate the periodic convolution $x * y$.

Solution

Convolutions with Kronecker delta are easy: $\delta_c[n] * f[n] = f[n-c]$. Note:

$$\begin{bmatrix}0 & 1 & 1 & 0 & 0 & 0 & -1 & 0\end{bmatrix} = \delta_1[n] + \delta_2[n] - \delta_{-2}[n]$$

Now if we let $\begin{bmatrix}\frac{1}{2} & \frac{1}{4} & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{4}\end{bmatrix} = f[n]$, then the answer is:

$$\delta_1 * f + \delta_2 * f - \delta_{-2} * f = f[n-1] + f[n-2] - f[n+2].$$

To finish the question, we need to write out answer as a vector.

$$\begin{aligned} f[n] &= \begin{bmatrix}\frac{1}{2} & \frac{1}{4} & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{4}\end{bmatrix} \\ f[n-1] &= \begin{bmatrix}\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{8}\end{bmatrix} \\ f[n-2] &= \begin{bmatrix}\frac{1}{8} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & 0 & 0 & 0\end{bmatrix} \\ f[n+2] &= \begin{bmatrix}\frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\end{bmatrix} \end{aligned}$$

Final answer: $\begin{bmatrix}\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{8} & 0 & -\frac{1}{4} & -\frac{1}{2} & -\frac{1}{8}\end{bmatrix}$

9. **[NOT TO HAND-IN]** [Periodic convolution]

Consider the folloing signals with period $N = 4$:

$$a = [1, 0, 1, -1], \qquad b = [2, i, 1+i, 3]$$

(e.g. $a[0] = 1, a[3] = -1, b[2] = 1+i$, etc.)

(a) Calculate the periodic convolution $a * b$ **by directly calculating the convolution sum**.

(b) Calculate the Fourier coefficients $\widehat{a}[k]$ and $\widehat{b}[k]$. Use this to compute the Fourier coefficients $\widehat{a * b}[k]$ for $a * b$ by using the convolution property of the Fourier transform.

(c) Find a signal $x[n]$ of period $N = 4$, such that $(a * x)[n] = b[n]$. (Hint: you may want to use the convolution property of the Fourier transform/inversion. Remember how we handle the circuit problem. This is similar.)

Solution

(a) :

$$(a*b)[n] = \sum_{m=0}^{3} a[m]b[n-m]$$

Thus, (noting $b[-1] = b[4-1] = b[3]; b[-2] = b[4-2] = b[2]; b[-3] = b[4-3] = b[1]$),

$$(a*b)[0] = \sum_{m=0}^{3} a[m]b[0-m] = 1\times 2 + 0\times 3 + 1\times(1+i) + (-1)\times i = 3$$

$$(a*b)[1] = \sum_{m=0}^{3} a[m]b[1-m] = 1\times i + 0\times 2 + 1\times 3 + (-1)\times(1+i) = 2$$

$$(a*b)[2] = \sum_{m=0}^{3} a[m]b[2-m] = 1\times(1+i) + 0\times i + 1\times 2 + (-1)\times 3 = i$$

$$(a*b)[3] = \sum_{m=0}^{3} a[m]b[3-m] = 1\times 3 + 0\times(1+i) + 1\times i + (-1)\times 2 = 1+i$$

So, $\underline{a*b = [3, 2, i, 1+i].}$

(b): Note that $e^{-i2\pi/4} = e^{i\pi/2} = -i$. Thus, $e^{-i2\pi/4kn} = (-i)^{kn}$. So,

$$\widehat{a}[k] = \frac{1}{4}\sum_{n=0}^{3} a[n]e^{-i2\pi/4kn} = \frac{1}{4}(1\times 1 + 0\times(-i)^k + 1\times(-i)^{2k} + (-1)\times(-i)^{3k})$$
$$= \frac{1}{4}(1 + (-1)^k - i^k)$$

Thus, $\underline{\widehat{a} = [1/4, -i/4, 3/4, i/4].}$

On the other hand,

$$\widehat{b}[k] = \frac{1}{4}\sum_{n=0}^{3} b[n]e^{-i2\pi/4kn} = \frac{1}{4}(2\times 1 + i\times(-i)^k + (1+i)\times(-i)^{2k} + 3\times(-i)^{3k})$$
$$= \frac{1}{4}(2 + (-1)^k i^{k+1} + (1+i)(-1)^k + 3i^k)$$

Thus, $\underline{\widehat{b} = [(6+2i)/4, (2+2i)/4, 0, -i].}$

Use convolution property for $N = 4$, to see

$$\widehat{a*b}[k] = 4\widehat{a}[k]\widehat{b}[k]$$

for $k = 0, \cdots, 3$. So,

$$\underline{\widehat{a * b} = [(6 + 2i)/4,\ (-2i + 2)/4,\ 0,\ 1]}$$

(c): Take Fourier transform:

$$\widehat{a * x}[k] = \widehat{b}[k]$$

By convolution property (for $N = 4$),

$$\widehat{a * x}[k] = 4\widehat{a}[k]\widehat{x}[k]$$

Thus, we see

$$4\widehat{a}[k]\widehat{x}[k] = \widehat{b}[k]$$

So,

$$\widehat{x}[0] = (6 + 2i)/4, \quad \widehat{x}[1] = (2i - 2)/4, \quad \widehat{x}[2] = 0, \quad \widehat{x}[3] = -1$$

i.e.

$$\widehat{x} = [(3 + i)/2, (-1 + i)/2, 0, -1].$$

To find $x[n]$, apply the inverse discrete Fourier transform:

$$\begin{aligned} x[n] &= \sum_{k=0}^{3} \widehat{x}[k] e^{i2\pi kn/4} \\ &= \frac{3+i}{2} + (\frac{-1+i}{2}) i^k + 0 + (-1) i^{3k} \\ &= \frac{3+i}{2} + \frac{-i^k + i^{k+1}}{2} + 0 + (-1)^{k+1} i^k \end{aligned}$$

Thus,

$$\underline{x = [i, 1 + i, 3, 2]}$$

10. **[NOT TO HAND-IN]** Consider two discrete signals, both length $N = 3$: $x[n] = [x[0], x[1], x[2]]$, and, $y[n] = [y[0], y[1], y[2]]$.

    (a) Write out the definition of $(x * y)[0]$.
    Think of $x[n]$ as a column vector $\vec{x}$.
    Recognize that the formula for $(x * y)[0]$ is a dot-product of some row vector $\vec{a}$ with $\vec{x}$.
    What are the components of vector $\vec{a}$?

(b) Repeat part(a) for $(x * y)[1]$ and $(x * y)[2]$.
Put all three row vectors into a matrix Y, so that,

$$(x * y) = Y\vec{x}.$$

(c) Use your answer to part(b) to compute $(x * y)$, where $x[n]$ and $y[n]$ are the following signals with length $N = 3$,

$$x = [2, -1, 1], \quad y = [-3, i, 1]$$

Solution

(a) Writing out the definition:

$$(x * y)[0] = \sum_{k=0}^{3-1} x[k]y[0-k] = y[0]x[0] + y[-1]x[1] + y[-2]x[2]$$

That is: $(x * y)[0] = \vec{a} \cdot \vec{x}$ for $\vec{a} = \left[\, y[0],\, y[-1],\, y[-2]\right]$

(b) Exactly the same argument:

$(x * y)[1] = \vec{b} \cdot \vec{x}$ for $\vec{b} = \left[\, y[1],\, y[0],\, y[-1]\right]$

$(x * y)[2] = \vec{c} \cdot \vec{x}$ for $\vec{c} = \left[\, y[2],\, y[1],\, y[0]\right]$

Put $\vec{a}$, $\vec{b}$, $\vec{c}$ as the rows of 3-by-3 matrix Y (in order).

(c) In this example,

$$Y = \begin{bmatrix} y[0] & y[-1] & y[-2] \\ y[1] & y[0] & y[-1] \\ y[2] & y[1] & y[0] \end{bmatrix} = \begin{bmatrix} -3 & 1 & i \\ i & -3 & 1 \\ 1 & i & -3 \end{bmatrix}$$

Normal matrix product gives:

$$Y\vec{x} = \begin{bmatrix} -5 + i \\ 4 + 2i \\ -1 - i \end{bmatrix}$$