Math 267, Section 202 : HW 7

Due **Monday, March 4th**. The topics covered will be **included** on your midterm, **Thursday, March 7th**

1. **(Not to be graded)** [Delta function and Fourier transform]
   (a) Let $f(t) = \delta(t-1) + 3 + \delta(1-2t) + e^{i2t}$. Find $\widehat{f}(\omega)$.
   (b) Let $\widehat{g}(\omega) = \delta(2\omega - 1) + 1 + \delta(2 - 2\omega) + e^{i\omega}$. Find $g(t)$.
   (c) Let $h(t) = u(t+1) + u(2t+1)$. Find the values $\widehat{h}(\pi)$ and $\widehat{h}(\pi/2)$.

2. Calculate the inverse FT.
   (a) $\widehat{m}(\omega) = \sin\left(3\omega - \frac{\pi}{4}\right)$
   (b) $\widehat{h}(\omega) = \frac{1}{-4\omega^2 + i\omega + 2}$
   (c) $\widehat{z}(\omega) = e^{-5(\omega+\pi)}u(\omega - 1)$

3. Let $f(t)$ be a continuous function on an interval $[a, b]$ with $a < 0 < b$. Consider the product $f(t)\delta(t)$. One can write $f(t)\delta(t) = C\delta(t)$ for some constant C. Your task is to determine C in terms of the function $f(t)$. For example, what is the value of C if $f(t) = \frac{1}{4+t^2}$?
   (Hint: This problem is very easy, once you understand the basic properties of the delta function.)

4. Recall the fact that for the unit step function $u(t)$, its Fourier transform is
$$\widehat{u}(\omega) = \frac{1}{i\omega} + \pi\delta(\omega).$$
   Suppose $f(t)$ has the Fourier transform
$$\widehat{f}(\omega) = \frac{1}{i\omega + 1}.$$
   Find the inverse Fourier transform of $\widehat{u}(\omega)\widehat{f}(\omega)$.

5. Compute the convolutions:
   (a) $\text{rect}(x) * \sin(x)$
   (b) $u(x) * u(x)$
   (Hint: Directly compute the corresponding integral.)

6. **(Not to be graded)** Consider the functions

$$f(t) = \begin{cases} -2, & -2 \le t < 0 \\ 1, & 0 \le t < 4 \\ 0, & \text{otherwise} \end{cases}$$

$$g(t) = \begin{cases} 1, & -1 \le t < 0 \\ 0, & \text{otherwise} \end{cases}$$

and $h(t) = (f * g)(\mathrm{t})$

(a) Find $h(t)$ and draw an accurate graph of this function on the interval $-4 \le t \le 5$. **Hint:** You should obtain a collection of straight line segments.

(b) Find $\widehat{h}(\omega)$. **Hint:** Use the convolution property, $\widehat{(f * g)}(\omega) = \widehat{f}(\omega)\widehat{g}(\omega)$.

7. For each real number a, denote $\delta_a(t) = \delta(t - a)$.

(a) Compute $(\delta_4 * \delta_{-3})(t)$, by taking the FT and then inverting the FT.

(b) Find a general formula for $(\delta_a * \delta_b)(t)$

(c) Compute the FT of $\cos(4t)\sin(-3t)$.
Simplify your answer so that there are no convolutions in the final expression.

8. Consider a circuit with *frequency response* $\widehat{H}(\omega)$ given by

$$\widehat{H}(\omega) = \frac{1}{-\omega^2 + 1}.$$

(The function $H(t)$ that is the inverse Fourier transform of $\widehat{H}(\omega)$, is called *impulse response.*)

Let the applied (input) voltage be

$$f_{in}(t) = u(t - 2).$$

What is the output voltage $f_{out}(t)$?

Recall that: $\widehat{f_{out}}(\omega) = \widehat{H}(\omega)\,\widehat{f_{in}}(\omega)$.