Math 267, Section 202 : HW 7

Due **Monday, March 4th**. The topics covered will be **included** on your midterm, **Thursday, March 7th**

1. **(Not to be graded)** [Delta function and Fourier transform]

   (a) Let $f(t) = \delta(t-1) + 3 + \delta(1-2t) + e^{i2t}$. Find $\widehat{f}(\omega)$.
   Solution

$$\begin{aligned}\mathcal{F}[f(t)](\omega) &= \mathcal{F}[\delta(t-1)](\omega) + \mathcal{F}[3[(\omega) + \mathcal{F}[\delta(2(1/2-t))](\omega) + \mathcal{F}[e^{i2t}](\omega)\\ &= e^{-i\omega} + 3 \times 2\pi\delta(\omega) + \frac{1}{2}e^{-i\omega/2} + 2\pi\delta(\omega-2)\\ &= \underline{e^{-i\omega} + 6\pi\delta(\omega) + \frac{1}{2}e^{-i\omega/2} + 2\pi\delta(\omega-2)}\end{aligned}$$

   (b) Let $\widehat{g}(\omega) = \delta(2\omega-1) + 1 + \delta(2-2\omega) + e^{i\omega}$. Find $g(t)$.
   Solution

$$\begin{aligned}\mathcal{F}^{-1}[g(t)](\omega) &= \mathcal{F}^{-1}[\delta(2(\omega-1/2))](t) + \mathcal{F}^{-1}[1[(t) + \mathcal{F}^{-1}[\delta(2(1-\omega))](t) + \mathcal{F}[e^{i\omega}](t)\\ &= \frac{1}{2}e^{it/2}\mathcal{F}^{-1}[\delta((\omega))](t/2) + \delta(t) + \frac{1}{2}e^{it}\mathcal{F}^{-1}[\delta(\omega)](t/2) + \delta(t+1)\\ &= \frac{1}{2}e^{it/2}\frac{1}{2\pi} + \delta(t) + \frac{1}{2}e^{it}\frac{1}{2\pi} + \delta(t+1)\\ &= \underline{e^{it/2}\frac{1}{4\pi} + \delta(t) + e^{it}\frac{1}{4\pi} + \delta(t+1)}\end{aligned}$$

   (c) Let $h(t) = u(t+1) + u(2t+1)$. Find the values $\widehat{h}(\pi)$ and $\widehat{h}(\pi/2)$.
   Solution Note that $\frac{d}{dt}u(t) = \delta(t)$, so

$$\begin{aligned}\frac{d}{dt}h(t) &= \delta(t+1) + 2\delta(2t+1)\\ &= \delta(t+1) + \delta(2(t+1/2)).\end{aligned}$$

   Now,

$$\mathcal{F}[\frac{d}{dt}h(t)](\omega) = i\omega\mathcal{F}[h(t)](\omega)$$

   on one hand, and on the other hand, we have

$$\mathcal{F}[\frac{d}{dt}h(t)](\omega) = \mathcal{F}[\delta(t+1) + \delta(2(t+1/2))](\omega) = e^{i\omega} + \frac{1}{2}e^{i\omega/2}$$

(here, of course, we used the time-shift and scaling property).
So, we get

$$i\omega\mathcal{F}[h(t)](\omega) = e^{i\omega} + \frac{1}{2}e^{i\omega/2}$$

In particular,

$$\begin{aligned} i\pi\widehat{h}(\pi) &= e^{i\pi} + \frac{1}{2}e^{i\pi/2} = -1 + i/2 \\ i\pi/2\widehat{h}(\pi/2) &= e^{i\pi/2} + \frac{1}{2}e^{i\pi/4} = i + \frac{\sqrt{2}}{4} + i\frac{\sqrt{2}}{4} \end{aligned}$$

Thus,

$$\begin{aligned} \underline{\widehat{h}(\pi) = \frac{-1 + i/2}{i\pi}} \\ \underline{\widehat{h}(\pi/2) = \frac{i(2+\sqrt{2}/2) + \sqrt{2}/2}{i\pi}}. \end{aligned}$$

2. Calculate the inverse FT.

   (a) $\widehat{m}(\omega) = \sin\left(3\omega - \frac{\pi}{4}\right)$

   Solution

   Method 1

$$\begin{aligned} \sin\left(3\omega - \frac{\pi}{4}\right) &= \frac{1}{2i}\left[e^{i(3\omega-\pi/4)} - e^{-i(3\omega-\pi/4)}\right] \\ &= \frac{1}{2i}\left[e^{-i\pi/4}e^{i3\omega} - e^{+i\pi/4}e^{-i3\omega}\right] \end{aligned}$$

   Thus, Fourier inversion gives (noting $\mathcal{F}^{-1}[e^{-ia\omega}](t) = \delta(t-a)$,

$$\underline{m(t) = \frac{1}{2i}e^{-i\pi/4}\delta(t+3) - \frac{1}{2i}e^{i\pi/4}\delta(t-3)}$$

Method 2

$$
\begin{aligned}
&\mathcal{F}^{-1}\left[\sin\left(3\omega - \frac{\pi}{4}\right)\right](t) \\
&= \mathcal{F}^{-1}\left[\sin\left(3(\omega - \frac{\pi}{12})\right)\right](t) \\
&= e^{it\pi/12}\mathcal{F}^{-1}\left[\sin(3\omega)\right](t) \quad \text{(frequency shifting)} \\
&= e^{it\pi/12}\frac{1}{3}F^{-1}\left[\sin(\omega)\right](t/3) \quad \text{(scaling)} \\
&= e^{it\pi/12}\frac{1}{3}F^{-1}\left[\frac{e^{i\omega} - e^{-i\omega}}{2i}\right](t/3) \\
&= e^{it\pi/12}\frac{1}{6i}F^{-1}\left[e^{i\omega} - e^{-i\omega}\right](t/3) \\
&= \frac{e^{it\pi/12}}{6i}[\delta(t/3+1) - \delta(t/3-1)] \quad \text{(time-shifting)} \\
&= \frac{e^{it\pi/12}}{6i}[\delta((t+3)/3) - \delta((t-3)/3)] \\
&= \frac{e^{it\pi/12}}{6i}[\delta((t+3)/3) - \delta((t-3)/3)] \\
&= \frac{e^{it\pi/12}}{6i}[3\delta((t+3)) - 3\delta((t-3))] \quad \text{(since } \delta(t/3) = 3\delta(t)) \\
&= \frac{e^{it\pi/12}}{2i}[\delta((t+3)) - \delta((t-3))] \\
&= \frac{e^{i(-3)\pi/12}}{2i}\delta((t+3)) - \frac{e^{i(3)\pi/12}}{2i}\delta((t-3)) \quad \text{(using problem 3)} \\
&= \underline{\frac{e^{-i\pi/4}}{2i}\delta((t+3)) - \frac{e^{i\pi/4}}{2i}[\delta((t-3))}
\end{aligned}
$$

(b) $\widehat{h}(\omega) = \frac{1}{-4\omega^2+i\omega+2}$

Solution

$$
\begin{aligned}
-4\omega^2 + i\omega + 2 &= 4(i\omega)^2 + i\omega + 2 \\
&= \left(i\omega - i\frac{\sqrt{31}}{8} + \frac{1}{8}\right)\left(i\omega + i\frac{\sqrt{31}}{8} + \frac{1}{8}\right) \\
&= \left(i(\omega - \frac{\sqrt{31}}{8}) + \frac{1}{8}\right)\left(i(\omega + \frac{\sqrt{31}}{8}) + \frac{1}{8}\right)
\end{aligned}
$$

Thus,

$$\frac{1}{-4\omega^2+i\omega+2} = \frac{1}{\left(i\omega - i\frac{\sqrt{31}}{8}+\frac{1}{8}\right)\left(i\omega + i\frac{\sqrt{31}}{8}+\frac{1}{8}\right)}$$
$$= \frac{-i4}{\sqrt{31}}\left(\frac{1}{\left(i\omega - i\frac{\sqrt{31}}{8}+\frac{1}{8}\right)} - \frac{1}{\left(i\omega + i\frac{\sqrt{31}}{8}+\frac{1}{8}\right)}\right)$$

Note that for $a>0$,

$$\mathcal{F}^{-1}\left[\frac{1}{i\omega - ic + a}\right](t) = \mathcal{F}^{-1}\left[\frac{1}{i(\omega-c)+a}\right](t) = e^{ict}e^{-at}u(t)$$

Therefore,

$$\mathcal{F}^{-1}\left[\frac{1}{-4\omega^2+i\omega+2}\right](t) = \underline{\frac{-4i}{\sqrt{31}}\left[e^{-(1/8-i\sqrt{31}/8)t}u(t) - e^{-(1/8+i\sqrt{31}/8)t}u(t)\right]}$$

(c) $\widehat{z}(\omega) = e^{-5(\omega+\pi)}u(\omega-1)$

Solution

$$\mathcal{F}^{-1}\left[e^{-5(\omega+\pi)}u(\omega-1)\right](t)$$
$$= \mathcal{F}^{-1}\left[e^{-5\pi-5}e^{-5(\omega-1)}u(\omega-1)\right](t)$$
$$= e^{-5\pi-5}\mathcal{F}^{-1}\left[e^{-5(\omega-1)}u(\omega-1)\right](t)$$
$$= e^{-5\pi-5}e^{it}\mathcal{F}^{-1}\left[e^{-5(\omega)}u(\omega)\right](t) \quad \text{(frequency shifting)}$$
$$= e^{-5\pi-5}e^{it}\frac{-1}{2\pi}\frac{1}{i(-t)+5} \quad \text{(duality)}$$
$$= \underline{\frac{e^{-5\pi-5}}{2\pi}e^{it}\frac{1}{i(-t)+5}}$$

3. Let $f(t)$ be a continuous function on an interval $[a,b]$ with $a<0<b$. Consider the product $f(t)\delta(t)$. One can write $f(t)\delta(t) = C\delta(t)$ for some constant C. Your task is to determine C in terms of the function $f(t)$. For example, what is the value of C if $f(t) = \frac{1}{4+t^2}$?

   (Hint: This problem is very easy, once you understand the basic properties of the delta function.)

   Solution :

$$\int_{\infty}^{\infty} f(t)\delta(t)dt = f(0) \quad (\text{ since } f \text{ is continuous at } t=0)$$
$$\int_{\infty}^{\infty} C\delta(t)dt = C\int_{\infty}^{\infty}\delta(t)dt = C.$$

   Comparing both, we get $C = f(0)$.

4. Recall the fact that for the unit step function $u(t)$, its Fourier transform is

$$\widehat{u}(\omega) = \frac{1}{i\omega} + \pi\delta(\omega).$$

Suppose $f(t)$ has the Fourier transform

$$\widehat{f}(\omega) = \frac{1}{i\omega + 1}.$$

Find the inverse Fourier transform of $\widehat{u}(\omega)\widehat{f}(\omega)$.

Solution :

$$\begin{aligned}
\widehat{u}(\omega)\widehat{f}(\omega) &= \frac{1}{i\omega+1}(\frac{1}{i\omega} + \pi\delta(\omega)) \\
&= \frac{1}{i\omega+1}\frac{1}{i\omega} + \pi\frac{1}{i\omega+1}\delta(\omega) \\
&= -\frac{1}{i\omega+1} + \frac{1}{i\omega} + \pi\frac{1}{i0+1}\delta(\omega) \qquad \text{(using problem 3)} \\
&= -\frac{1}{i\omega+1} + [\frac{1}{i\omega} + \pi\delta(\omega)]
\end{aligned}$$

Thus, taking Fourier inversion we get

$$\underline{-e^{-t}u(t) + u(t)}$$

5. Compute the convolutions:

   (a) $\text{rect}(x) * \sin(x)$

   (b) $u(x) * u(x)$
   (Hint: Directly compute the corresponding integral.)

Solution

(a) Using the integral definition:

$$\begin{aligned}
\text{rect}(x) * \sin(x) &= \int_{-\infty}^{\infty} \text{rect}(s)\sin(t-s)\,ds \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 \cdot \sin(t-s)\,ds \\
&= \cos(t-s)|_{s=-\frac{1}{2}}^{\frac{1}{2}} = \cos(t-\frac{1}{2}) - \cos(t+\frac{1}{2})
\end{aligned}$$

It is also possible to answer the question using Fourier transform.

(b) Using the integral definition:

$$\begin{aligned} u(x) * u(x) &= \int_{-\infty}^{\infty} u(s)\, u(t-s)\, ds \\ &= \int_{0}^{\infty} 1 \cdot u(t-s)\, ds \\ &= \begin{cases} \displaystyle\int_0^t 1 \cdot 1\, ds = t & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases} \end{aligned}$$

6. **(Not to be graded)** Consider the functions

$$\begin{aligned} f(t) &= \begin{cases} -2, & -2 \le t < 0 \\ 1, & 0 \le t < 4 \\ 0, & \text{otherwise} \end{cases} \\ g(t) &= \begin{cases} 1, & -1 \le t < 0 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

and $h(t) = (f * g)(\mathrm{t})$

(a) Find $h(t)$ and draw an accurate graph of this function on the interval $-4 \le t \le 5$. **Hint:** You should obtain a collection of straight line segments.

Solution : Notice that $f(t) = -2\mathrm{rect}((t+1)/2) + \mathrm{rect}((t-2)/4)$ and $g(t) = \mathrm{rect}(t + 1/2)$. Letting $f_1(t) = \mathrm{rect}((t+1)/2)$, $f_2(t) = \mathrm{rect}((t-2)/4)$, we see

$$(f * g)(t) = ((-2f_1 + f_2) * g)(t) = -2(f_1 * g)(t) + (f_2 * g)(t)$$

Now,

$$\begin{aligned} (f_1 * g)(t) &= \int_{\infty}^{\infty} \mathrm{rect}((s+1)/2)\mathrm{rect}(t - s + 1/2) ds \\ &= \int_{-2}^{0} \mathrm{rect}(t + 1/2 - s) ds \\ &= \begin{cases} 0 & \text{for } t + 1/2 < -2 - 1/2, \text{ i.e. for } t < -3 \text{ ,} \\ t-3 & \text{for } -2 - 1/2 < t + 1/2 < -2 + 1/2, \text{ i.e. for } -3 < t < -2, \\ 1 & \text{for } -2 + 1/2 < t + 1/2 < 0 - 1/2, \text{ i.e. for } -2 < t < -1 \\ -t & \text{for } 0 - 1/2 < t + 1/2 < 0 + 1/2, \text{ i.e. for } -1 < t < 0, \\ 0 & \text{for } 0 + 1/2 < t + 1/2, \text{ i.e. for } t > 0. \end{cases} \end{aligned}$$

Also,

$$
\begin{aligned}
(f_2 * g)(t) &= \int_{\infty}^{\infty} \mathrm{rect}((s-2)/4)\mathrm{rect}(t-s+1/2)ds \\
&= \int_0^4 \mathrm{rect}(t+1/2-s)ds \\
&= \begin{cases} 0 & \text{for } t+1/2 < 0-1/2, \text{ i.e. for } t < -1, \\ t+1 & \text{for } 0-1/2 < t+1/2 < 0+1/2, \text{ i.e. for } -1 < t < 0, \\ 1 & \text{for } 0+1/2 < t+1/2 < 4-1/2, \text{ i.e. for } 0 < t < 3 \\ 4-t & \text{for } 4-1/2 < t+1/2 < 4+1/2, \text{ i.e. for } 3 < t < 4, \\ 0 & \text{for } 4+1/2 < t+1/2, \text{ i.e. for } t > 4. \end{cases}
\end{aligned}
$$

Combining these two, we get

$$
(f * g)(t) = \begin{cases} 0 & \text{for } t < -3, \\ -2(t-3) & \text{for } -3 < t < -2, \\ -2 & \text{for } -2 < t < -1 \\ -2(-t) + t + 1 & \text{for } -1 < t < 0, \\ 1 & \text{for } 0 < t < 3, \\ 4-t & \text{for } 3 < t < 4, \\ 0 & \text{for } 4 < t. \end{cases}
$$

(b) Find $\widehat{h}(\omega)$. **Hint:** Use the convolution property, $\widehat{(f * g)}(\omega) = \widehat{f}(\omega)\widehat{g}(\omega)$.

Solution :

From convolution property,

$$\widehat{h}(\omega) = \widehat{f}(\omega)\widehat{g}(\omega)$$

Here,

$$
\begin{aligned}
\widehat{f}(\omega) &= \mathcal{F}\left[-2\mathrm{rect}((t+1)/2) + \mathrm{rect}((t-2)/4)\right](\omega) \\
&= -2e^{i\omega}2\mathrm{sinc}(2\omega) + e^{-i\omega}\mathrm{sinc}(4\omega)
\end{aligned}
$$

and

$$\widehat{g}(\omega) = \mathcal{F}\left[\mathrm{rect}(t+1/2)\right](\omega) = e^{i\omega/2}\mathrm{sinc}(\omega)$$

Finally,

$$
\begin{aligned}
\widehat{h}(\omega) &= (-2e^{i\omega}2\mathrm{sinc}(2\omega) + e^{-i\omega}\mathrm{sinc}(4\omega))e^{i\omega/2}\mathrm{sinc}(\omega) \\
&= \underline{-4e^{i3\omega/2}\mathrm{sinc}(2\omega)\mathrm{sinc}(\omega) + e^{-i\omega/2}\mathrm{sinc}(4\omega)\mathrm{sinc}(\omega)}
\end{aligned}
$$

7. For each real number a, denote $\delta_a(t) = \delta(t-a)$.

   (a) Compute $(\delta_4 * \delta_{-3})(t)$, by taking the FT and then inverting the FT.

   (b) Find a general formula for $(\delta_a * \delta_b)(t)$

   (c) Compute the FT of $\cos(4t)\sin(-3t)$.
   Simplify your answer so that there are no convolutions in the final expression.

Solution

(a)

$$\begin{aligned}\mathcal{F}[\delta_4 * \delta_{-3}](\omega) &= \mathcal{F}[\delta_4] \cdot \mathcal{F}[\delta_{-3}] \\ &= e^{-i\omega\,4} \cdot e^{-i\omega(-3)} \\ &= e^{-i\omega\,1} = \mathcal{F}[\delta_1(t)](\omega)\end{aligned}$$

(b) The same argument as part(a) gives: $\delta_a * \delta_b = \delta_{a+b}(t)$

(c) First, individually:

$$\mathcal{F}[\cos(4t)] = \mathcal{F}\left[\frac{e^{i4\omega}}{2} + \frac{e^{-i4\omega}}{2}\right] = \pi\left(\delta_4(\omega) + \delta_{-4}(\omega)\right)$$

$$\mathcal{F}[\sin(-3t)] = \mathcal{F}\left[\frac{e^{-i3\omega}}{2i} - \frac{e^{+i3\omega}}{2i}\right] = i\pi\left(\delta_3(\omega) - \delta_{-3}(\omega)\right)$$

Using the multiplication rule, and part(b):

$$\begin{aligned}\mathcal{F}[\cos(4t)\sin(-3t)] &= \frac{1}{2\pi}\left(\pi\left(\delta_4(\omega) + \delta_{-4}(\omega)\right)\right) * \left(i\pi\left(\delta_3(\omega) - \delta_{-3}(\omega)\right)\right) \\ &= \frac{i\pi}{2}\left(\delta_7 - \delta_1 + \delta_{-1} - \delta_{-7}\right)\end{aligned}$$

You can also expand $\cos(4t)\sin(-3t)$ as complex exponentials, and then transform.

8. Consider a circuit with *frequency response* $\widehat{H}(\omega)$ given by

$$\widehat{H}(\omega) = \frac{1}{-\omega^2 + 1}.$$

(The function $H(t)$ that is the inverse Fourier transform of $\widehat{H}(\omega)$, is called *impulse response*.)

Let the applied (input) voltage be

$$f_{in}(t) = u(t-2).$$

What is the output voltage $f_{out}(t)$?

Recall that: $\widehat{f_{out}}(\omega) = \widehat{H}(\omega)\,\widehat{f_{in}}(\omega)$.

There was an error in the previous solution, in the partial fraction of $\frac{1}{-\omega^2+1}$. The error is fixed in this new solution.

Solution

First,

$$\frac{1}{-\omega^2+1} = \frac{1}{2}\left[\frac{1}{\omega+1} - \frac{1}{\omega-1}\right]$$

Notice that $\mathcal{F}\left[u(t-2)\right] = e^{-i2\omega}\left(\frac{1}{i\omega} + \pi\delta(\omega)\right)$.

Thus,

$$\begin{aligned}\widehat{f_{out}}(\omega) &= \widehat{H}(\omega)\,\widehat{f_{in}}(\omega)\\ &= \frac{1}{2}\left[\frac{1}{\omega+1} - \frac{1}{\omega-1}\right] e^{-i2\omega}\left(\frac{1}{i\omega} + \pi\delta(\omega)\right)\\ &= \frac{1}{2}e^{-i2\omega}\left[\frac{1}{\omega+1}\left(\frac{1}{i\omega} + \pi\delta(\omega)\right) - \frac{1}{\omega-1}\left(\frac{1}{i\omega} + \pi\delta(\omega)\right)\right]\end{aligned}$$

Here,

$$\begin{aligned}\frac{1}{\omega+1}\left(\frac{1}{i\omega} + \pi\delta(\omega)\right) &= \frac{1}{\omega+1}\frac{1}{i\omega} + \pi\frac{1}{\omega+1}\;\delta(\omega) = \frac{1}{\omega+1}\frac{1}{i\omega} + \pi\frac{1}{0+1}\;\delta(\omega)\\ &= \frac{1}{\omega+1}\frac{1}{i\omega} + \pi\delta(\omega) = -\frac{1}{i(\omega+1)} + \frac{1}{i\omega} + \pi\delta(\omega)\\ &= -\frac{1}{i(\omega+1)} + \left[\frac{1}{i\omega} + \pi\delta(\omega)\right]\end{aligned}$$

Also,

$$\begin{aligned}\frac{1}{\omega-1}\left(\frac{1}{i\omega} + \pi\delta(\omega)\right) &= \frac{1}{\omega-1}\frac{1}{i\omega} + \pi\frac{1}{\omega-1}\;\delta(\omega) = \frac{1}{\omega-1}\frac{1}{i\omega} + \pi\frac{1}{0-1}\;\delta(\omega)\\ &= \frac{1}{\omega-1}\frac{1}{i\omega} - \pi\delta(\omega) = \frac{1}{i(\omega-1)} - \frac{1}{i\omega} - pi\delta(\omega)\\ &= \frac{1}{i(\omega-1)} - \left[\frac{1}{i\omega} + \pi\delta(\omega)\right]\end{aligned}$$

As a result, we have

$$\begin{aligned}\widehat{f}_{out}(\omega) &= \frac{1}{2}e^{-i2\omega}\left[-\frac{1}{i(\omega+1)} + \left[\frac{1}{i\omega} + \pi\delta(\omega)\right] - \frac{1}{i(\omega-1)} + \left[\frac{1}{i\omega} + \pi\delta(\omega)\right]\right]\\ &= \frac{1}{2}e^{-i2\omega}\left[-\frac{1}{i(\omega+1)} - \frac{1}{i(\omega-1)} + 2\left[\frac{1}{i\omega} + \pi\delta(\omega)\right]\right]\end{aligned}$$

Here, to realize $\frac{1}{i(\omega \pm 1)}$ as a Fourier transform, we express them as

$$\begin{aligned} \frac{1}{i(\omega \pm 1)} &= \left(\frac{1}{i(\omega \pm 1)} + \pi\delta(\omega \pm 1)\right) - \pi\delta(\omega \pm 1) \\ &= \mathcal{F}\left[e^{\mp it}u(t)\right](\omega) - \pi\mathcal{F}\left[\frac{1}{2\pi}e^{\mp it}\right](\omega) \end{aligned}$$

(Here used frequency shifting property.)

So the Fourier inversion of $\widehat{f}_{out}$ is (after using time-shifting, taking care of the factor $e^{-i2\omega}$),

$$\frac{1}{2}\left[-e^{-i(t-2)}u(t-2) + \frac{1}{2}e^{-i(t-2)} - e^{i(t-2)}u(t-2) + \frac{1}{2}e^{i(t-2)} + 2u(t-2)\right]$$

We can simplify this as

$$\underline{-\cos(t-2)u(t-2) + \cos(t-2) + u(t-2)}$$

9. (There was an error in the previous solution for Problem 8. The previous solution works for this problem, with different $\widehat{H}(\omega)$.) Consider a circuit with *frequency response* $\widehat{H}(\omega)$ given by

$$\widehat{H}(\omega) = \frac{1}{-\omega^2 - 1}.$$

(The function $H(t)$ that is the inverse Fourier transform of $\widehat{H}(\omega)$, is called *impulse response*.)

Let the applied (input) voltage be

$$f_{in}(t) = u(t-2).$$

What is the output voltage $f_{out}(t)$?

Recall that: $\widehat{f_{out}}(\omega) = \widehat{H}(\omega)\,\widehat{f_{in}}(\omega)$.

Solution

Method 1:

First,

$$\frac{1}{-\omega^2 - 1} = \frac{1}{2}\left[\frac{1}{i\omega - 1} - \frac{1}{i\omega + 1}\right]$$

Notice that $\mathcal{F}[u(t-2)] = e^{-i2\omega}\left(\frac{1}{i\omega} + \pi\delta(\omega)\right)$. Thus,

$$
\begin{aligned}
\widehat{f_{out}}(\omega) &= \widehat{H}(\omega)\,\widehat{f_{in}}(\omega) \\
&= \frac{1}{2}\left[\frac{1}{i\omega-1} - \frac{1}{i\omega+1}\right] e^{-i2\omega}\left(\frac{1}{i\omega} + \pi\delta(\omega)\right) \\
&= \frac{1}{2}e^{-i2\omega}\left[\frac{1}{i\omega-1}\left(\frac{1}{i\omega} + \pi\delta(\omega)\right) - \frac{1}{i\omega+1}\left(\frac{1}{i\omega} + \pi\delta(\omega)\right)\right]
\end{aligned}
$$

Here,

$$
\begin{aligned}
\frac{1}{i\omega-1}\left(\frac{1}{i\omega} + \pi\delta(\omega)\right) &= \frac{1}{i\omega-1}\frac{1}{i\omega} + \pi\frac{1}{i\omega-1}\,\delta(\omega) = \frac{1}{i\omega-1}\frac{1}{i\omega} + \pi\frac{1}{i0-1}\,\delta(\omega) \\
&= \frac{1}{i\omega-1}\frac{1}{i\omega} - \pi\delta(\omega) = \frac{1}{i\omega-1} - \frac{1}{i\omega} - \pi\delta(\omega) \\
&= \frac{1}{i\omega-1} - \left[\frac{1}{i\omega} + \pi\delta(\omega)\right]
\end{aligned}
$$

Similar computation gives,

$$
\frac{1}{i\omega+1}\left(\frac{1}{i\omega} + \pi\delta(\omega)\right) = -\frac{1}{i\omega+1} + \left[\frac{1}{i\omega} + \pi\delta(\omega)\right]
$$

Therefore,

$$
\begin{aligned}
f_{out}(t) &= \mathcal{F}^{-1}\left[\widehat{f}_{out}(\omega)\right] \\
&= \mathcal{F}^{-1}\left[\frac{1}{2}e^{-i2\omega}\left(\frac{1}{i\omega-1} - \left(\frac{1}{i\omega} + \pi\delta(\omega)\right) + \frac{1}{i\omega+1} - \left(\frac{1}{i\omega} + \pi\delta(\omega)\right)\right)\right](t) \\
&= \frac{1}{2}\mathcal{F}^{-1}\left[\frac{1}{i\omega-1} - \left(\frac{1}{i\omega} + \pi\delta(\omega)\right) + \frac{1}{i\omega+1} - \left(\frac{1}{i\omega} + \pi\delta(\omega)\right)\right](t-2) \\
&= \frac{1}{2}\left[-e^{t-2}u(-t+2) - u(t-2) + e^{-(t-2)}u(t-2) - u(t-2)\right] \\
&= \underline{\frac{1}{2}\left[-2u(t-2) - e^{t-2}u(2-t) + e^{-(t-2)}u(t-2)\right]}
\end{aligned}
$$

Method 2:

First,

$$
\widehat{H}(\omega) = \frac{1}{-\omega^2-1} = \frac{1}{2}\left[\frac{1}{i\omega-1} - \frac{1}{i\omega+1}\right]
$$

$$
\begin{aligned}
\widehat{f_{out}}(\omega) &= \widehat{H}(\omega)\,\widehat{f_{in}}(\omega) \\
&= \frac{1}{2}\left[\frac{1}{i\omega-1}\widehat{f_{in}}(\omega) - \frac{1}{i\omega+1}\widehat{f_{in}}\right](\omega)
\end{aligned}
$$

Notice that $\mathcal{F}^{-1}\left[\frac{1}{i\omega-1}\right](t) = -e^t u(-t)$ and $\mathcal{F}^{-1}\left[\frac{1}{i\omega+1})\right](t) = e^{-t}u(t)$. Thus for $g_1(t) = -e^t u(-t)$ and $g_2(t) = e^{-t}u(t)$, we have

$$\mathcal{F}^{-1}\left[\frac{1}{i\omega-1}\widehat{f}_{in}(\omega)\right](t) = (g_1 * f_{in})(t), \qquad \mathcal{F}^{-1}\left[\frac{1}{i\omega+1}\widehat{f}_{in}(\omega)\right](t) = (g_2 * f_{in})(t).$$

Here, recalling $f_{in}(t) = u(t-2)$,

$$\begin{aligned}(g_1 * f_{in})(t) &= \int_{-\infty}^{\infty} -e^s u(-s) f_{in}(t-s)ds \\ &= \int_{-\infty}^{\infty} -e^s u(-s) u(t-s-2)ds \\ &= \int_{-\infty}^{\infty} -e^s u(-s) u(t-2-s)ds \quad = \begin{cases} \int_{-\infty}^{0} -e^s ds & t-2>0\ , \\ \int_{-\infty}^{t-2} -e^s ds & t-2<0. \end{cases} \\ &= \begin{cases} -1 & t-2>0\ , \\ -e^{t-2} & t-2<0. \end{cases} \\ &= -u(t-2) - e^{t-2}u(2-t)\end{aligned}$$

and

$$\begin{aligned}(g_2 * f_{in})(t) &= \int_{-\infty}^{\infty} e^{-s} u(s) f_{in}(t-s)ds \\ &= \int_{-\infty}^{\infty} e^{-s} u(s) u(t-s-2)ds \\ &= \int_{-\infty}^{\infty} e^{-s} u(s) u(t-2-s)ds \quad = \begin{cases} 0 & t-2<0\ , \\ \int_{0}^{t-2} e^{-s} ds & t-2>0. \end{cases} \\ &= \begin{cases} 0 & t-2<0\ , \\ -e^{-(t-2)}+1 & t-2>0. \end{cases} \\ &= (1-e^{-(t-2)})u(t-2).\end{aligned}$$

Finally,

$$\begin{aligned}f_{out} &= \frac{1}{2}\mathcal{F}^{-1}\left[\frac{1}{i\omega-1}\widehat{f_{in}}(\omega) - \frac{1}{i\omega+1}\widehat{f_{in}}(\omega)\right](t) \\ &= \frac{1}{2}\left((g_1 * f_{in})(t) - (g_2 * f_{in})(t)\right) \\ &= \frac{1}{2}\Big(-u(t-2) - e^{t-2}u(2-t) - (1-e^{-(t-2)})u(t-2)\Big) \\ &= \underline{\frac{1}{2}\Big(-2u(t-2) - e^{t-2}u(2-t) + e^{-(t-2)}u(t-2)\Big)}\end{aligned}$$