Math 267, Section 202 : HW 6

Due **Monday, February 25th**.

1. **(Scaling, time-shift, duality, differentiation)**

   (a) Find Fourier transform of

   $$f(t) = \begin{cases} t+1, & -1 \le t \le -1/2; \\ -t, & -1/2 \le t \le 0; \\ 0, & \text{otherwise.} \end{cases}$$

   (**Hint:** This is similar to one of class examples about differentiation rule for Fourier transform.)

   (b) Find Fourier transform of

   $$f(t) = \begin{cases} t+2, & -2 \le t \le -1; \\ -t, & -1 \le t \le 0; \\ 0, & \text{otherwise.} \end{cases}$$

   (**Hint:** Use (a) and scaling property of Fourier transform.)

   (c) Let $f(t) = e^{-|t|}$.

   i. Find $\widehat{f}(\omega)$. (**Hint:** this is a class example. You can use the result for $e^{-t}u(t)$ and apply properties of Fourier transform: here time-reversal property is relevant.)

   ii. Use part (i) and the duality property to find the Fourier transform $\widehat{g}(\omega)$ of the function

   $$g(t) = \frac{1}{\pi}\frac{1}{1+t^2}$$

2. Find the inverse Fourier transform of the following functions. Namely, for the given Fourier transform function, find the original function.

   (a) $\widehat{g}(\omega) = \frac{1}{2+i\omega} - \frac{1}{3+i\omega}$

   (b) $\widehat{f}(\omega) = e^{-i2\omega}\mathrm{sinc}(3\omega)$

3. Find the inverse Fourier transforms.

   Use only basic examples, shortcuts and properties discussed in lecture.

   (a) $\widehat{f}(\omega) = \frac{2}{(i\omega+4)(i\omega-3)(i\omega+5)}$

   (b) $\widehat{g}(\omega) = \frac{1}{-2\omega^2+2i\omega+1}$

(c) $\widehat{h}(\omega) = \cos(\omega)\text{sinc}(\omega)$.
(Hint: Express $\cos(\omega)$ in terms of complex exponentials.)

4. Evaluate each integral. (Hint: You may want to recall the definition of Fourier transfrom or Fourier inverse transform, their properties, and basic examples.)

   (a) $\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2}{\omega^2+1}\, d\omega$

   (b) $\int_{-\infty}^{\infty} \text{sinc}(4\omega)\, e^{-i4\omega}\, d\omega$

5. **(Frequency Shifting)**

   (a) Show that if $g(t) = e^{i\omega_0 t} f(t)$, then $\widehat{g}(\omega) = \widehat{f}(\omega-\omega_0)$. Also, show that if $\widehat{g}(\omega) = \widehat{f}(\omega - \omega_0)$, then $g(t) = e^{i\omega_0 t} f(t)$. (Hint: Use the definition (the integral expression) of Fourier transform and Fourier inversion.)

   (b) For a function $h_1(t)$, suppose $\widehat{h}_1(\omega) = \text{sinc}(\frac{\omega}{2} - 2)$. Find $h_1(t)$. (Hint: use (a).)

   (c) For a function $h_2(t)$, suppose $\widehat{h}_2(\omega) = \text{sinc}(\frac{\omega}{2} - 2) + 2\,\text{sinc}(\frac{\omega}{2} + 2)$. Find $h_2(t)$. (Hint: use (a) and the linearity of Fourier transform/ inversion.)

6. (**Differentiation in frequency**)

   (a) Prove the following:

   $$\text{if } g(t) = tf(t) \text{ then } \widehat{g}(\omega) = i\tfrac{d}{d\omega}\widehat{f}(\omega)$$

   (**Hint:** differentiate the definition (I mean, the integral) of $\widehat{f}(\omega)$ with respect to ω: i.e.

   $$\frac{d}{d\omega}\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\frac{d}{d\omega}e^{-it\omega}dt. \quad)$$

   (b) Use (a) to show

   $$\text{if } \widehat{g}(\omega) = \tfrac{d}{d\omega}\widehat{f}(\omega), \text{ then } g(t) = -itf(t)$$

   (c) Using the frequency differentiation property in part (a), compute the Fourier transform of:

   (i) $f(t) = t\,\text{rect}(t)$

   (ii) $g(t) = t^2 e^{-3t} u(t)$ (**Hint:** you can apply the frequency differentiation property twice.)

   (d) [Fourier inversion] For a real nonzero constant a, find the function $g(t)$ if

   $$\widehat{g}(\omega) = \frac{1}{(i\omega + a)^2}$$

   (**Hint:** You can use (b). Can you express $\widehat{g}(\omega)$ as a ω-derivative of certain function?)

7. **(Optional. Not to be graded.)** It is known that

$$\mathcal{F}\left[e^{-\frac{x^2}{2}}\right](\omega) = \sqrt{2\pi}\; e^{-\frac{\omega^2}{2}}$$

Use this fact to calculate the Fourier transform of $m(x) = x\, e^{-\frac{x^2}{2}}$.

8. **(Optional. Not to be graded.) (RLC circuit)** Consider the ODE for RLC circuit:

$$LCy''(t) + RCy'(t) + y(t) = x(t)$$

   (a) Let $R = 4$, $L = 3$, $C = 1$ and $\widehat{x}(\omega) = 1$. Find $y(t)$ using Fourier transform method.

   (b) Let $R = 2$, $L = 1$, $C = 1$ and $\widehat{x}(\omega) = 1$. Find $y(t)$ using Fourier transform method.

   (c) Let $R = 4$, $L = 3$, $C = 1$ and $x(t) = u(t)e^{-2t}$. Find $y(t)$ using Fourier transform method.

9. **(Optional. Not to be graded.)** What is the inverse Fourier transform of $\widehat{q}(\omega) = e^{+i\omega}\frac{d}{d\omega}\text{sinc}(\omega + 1)$?

   Answer using properties of Fourier transform / Fourier inversion. (Hint: You may want to use the differentiation in frequency property in one of the previous problems. Alternatively, duality can be useful here.)