Math 267, Section 202 : HW 6

Due **Monday, February 25th**.

1. **(Scaling, time-shift, duality, differentiation)**

   (a) Find Fourier transform of

$$f(t) = \begin{cases} t+1, & -1 \le t \le -1/2; \\ -t, & -1/2 \le t \le 0; \\ 0, & \text{otherwise.} \end{cases}$$

(**Hint:** This is similar to one of class examples about differentiation rule for Fourier transform.)

Solution : Note that

$$\frac{d}{dt} f(t) = \text{rect}(2(t+3/4)) - \text{rect}(2(t+1/4)).$$

[For this, do first the scaling of the class example (scale by 1/2) and do the appropriate time-shift (by -1/4).]

Therefore, the Fourier transform

$\mathcal{F}[\frac{d}{dt} f(t)](\omega)$

$= \mathcal{F}[\text{rect}(2(t+3/4)) - \text{rect}(2(t+1/4))](\omega)$

$= \mathcal{F}[\text{rect}(2(t+3/4))](\omega) - \mathcal{F}[\text{rect}(2(t+1/4))](\omega)$ (by linearity of F.T.)

$= e^{i\omega 3/4}\mathcal{F}[\text{rect}(2t)](\omega) - e^{i\omega/4}\mathcal{F}[\text{rect}(2t)](\omega)$

(by time-shift property: practically it can be better to do this step first before handling scaling.)

$= [e^{i\omega 3/4} - e^{i\omega/4}]\mathcal{F}[\text{rect}(2t)](\omega)$

$= [e^{i\omega 3/4} - e^{i\omega/4}]\frac{1}{2}\mathcal{F}[\text{rect}(t)](\omega/2)$ (by scaling property)

$= \frac{1}{2}[e^{i\omega 3/4} - e^{i\omega/4}]\text{sinc}(\omega/4)$ (see $\omega/4$ in sinc instead of $\omega/2$!)

$= \frac{e^{i\omega/2}}{2}[e^{i\omega/4} - e^{-i\omega/4}]\text{sinc}(\omega/4)$

$= ie^{i\omega/2}\sin(\omega/4)\text{sinc}(\omega/4)$

But, on the other hand $\mathcal{F}[\frac{d}{dt}f(t)](\omega) = i\omega\mathcal{F}[f(t)](\omega)$ by the differentiation rule.

Therefore, for $\omega \neq 0$, we see

$$\begin{aligned}\mathcal{F}[f(t)](\omega) &= \frac{1}{i\omega} i e^{i\omega/2}\sin(\omega/4)\text{sinc}(\omega/4)\\ &= \frac{e^{i\omega/2}}{4}\text{sinc}(\omega/4)\text{sinc}(\omega/4)\\ &= \frac{e^{i\omega/2}}{4}[\text{sinc}(\omega/4)]^2\end{aligned}$$

For $\omega = 0$, we can directly compute the integral

$$\mathcal{F}[f(t)](0) = \int_{-\infty}^{\infty} f(t)e^{-i0t}dt = \int_{-\infty}^{\infty} f(t)dt = 1/4$$

(Note that when $\omega = 0$, $\frac{e^{i0/2}}{4}[\text{sinc}(0/4)]^2 = 1/4$.) Therefore, we have

$$\underline{\mathcal{F}[f(t)](\omega) = \frac{e^{i\omega/2}}{4}[\text{sinc}(\omega/4)]^2.}$$

(b) Find Fourier transform of

$$f(t) = \begin{cases} t+2, & -2 \leq t \leq -1; \\ -t, & -1 \leq t \leq 0; \\ 0, & \text{otherwise.} \end{cases}$$

(**Hint:** Use (a) and scaling property of Fourier transform.)

Solution

Let $f_1(t)$ denote the function $f(t)$ in part (a). Now for $f(t)$ in this part (b), we see that

$$f(t) = 2f_1(t/2).$$

Therefore,

$$\mathcal{F}[f(t)](\omega) = 2\times 2\mathcal{F}[f(_1(t)](2\omega)$$

So, we have

$$\begin{aligned}\widehat{f}(\omega) &= 4\frac{e^{i\omega}}{4}[\text{sinc}(\omega/2)]^2\\ &= \underline{e^{i\omega}[\text{sinc}(\omega/2)]^2}\end{aligned}$$

Remark: In fact, it can be easier to do this part (b) first and to use this to do part (a). The function in part (b) is nothing but a time-shift of the class example and the function in part (a) is the scaled function of the function in part (b) by scale factor 1/2.

(c) Let $f(t) = e^{-|t|}$.

i. Find $\widehat{f}(\omega)$. (**Hint:** this is a class example. You can use the result for $e^{-t}u(t)$ and apply properties of Fourier transform: here time-reversal property is relevant.)

Solution Let

$$f_0(t) = e^{-t}u(t)$$

Note that $\mathcal{F}[f_0(t)](\omega) = \frac{1}{i\omega+1}$ (this is one of the standard example given in the class). Now, we can write

$$f(t) = f_0(t) + f_0(-t).$$

Therefore,

$$\begin{aligned}
\mathcal{F}[f(t)](\omega) &= \mathcal{F}[f_0(t)](\omega) + \mathcal{F}(f_0(-t)](\omega) \\
&= \mathcal{F}[f_0(t)](\omega) + \mathcal{F}(f_0(t)](-\omega) \\
&\qquad \text{(used time-reversal property } \mathcal{F}[g(-t)](\omega) = \mathcal{F}[g(t)](-\omega).)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{F}[f(t)](\omega) &= \frac{1}{i\omega+1} + \frac{1}{-i\omega+1} \\
&= \underline{\frac{2}{\omega^2+1}}
\end{aligned}$$

ii. Use part (i) and the duality property to find the Fourier transform $\widehat{g}(\omega)$ of the function

$$g(t) = \frac{1}{\pi}\frac{1}{1+t^2}$$

Solution From (i), we sees that $\mathcal{F}[\frac{1}{2\pi}e^{-|t|}](\omega) = \frac{1}{\pi}\frac{1}{1+t^2}$. Thus by duality, $\mathcal{F}[\frac{1}{\pi}\frac{1}{1+t^2}](\omega) = 2\pi\frac{1}{2\pi}e^{-|-\omega|} = \underline{e^{-|\omega|}}$.

2. Find the inverse Fourier transform of the following functions. Namely, for the given Fourier transform function, find the original function.

(a) $\widehat{g}(\omega) = \frac{1}{2+i\omega} - \frac{1}{3+i\omega}$

Solution :

$$\begin{aligned}
g(t) &= \mathcal{F}^{-1}[\frac{1}{2+i\omega} - \frac{1}{3+i\omega}](t) \\
&= \mathcal{F}^{-1}[\frac{1}{2+i\omega}](t) - \mathcal{F}^{-1}[\frac{1}{3+i\omega}](t) \\
&= \underline{e^{-2t}u(t) - e^{-3t}u(t)}.
\end{aligned}$$

(b) $\widehat{f}(\omega) = e^{-i2\omega}\text{sinc}(3\omega)$

Solution :

$$\begin{aligned} f(t) &= \mathcal{F}^{-1}[e^{-i2\omega}\text{sinc}(3\omega)](t) \\ &= \mathcal{F}^{-1}[\text{sinc}(3\omega)](t-2) \quad \text{(time-shifting)} \\ &= \mathcal{F}^{-1}[\text{sinc}(6\omega/2)](t-2) \\ &= \frac{1}{6}\mathcal{F}^{-1}[\text{sinc}(\omega/2)](\frac{t-2}{6}) \quad \text{(scaling)} \\ &= \underline{\frac{1}{6}\text{rect}(\frac{t-2}{6})}. \end{aligned}$$

3. Find the inverse Fourier transforms.

   Use only basic examples, shortcuts and properties discussed in lecture.

   (a) $\widehat{f}(\omega) = \frac{2}{(i\omega+4)(i\omega-3)(i\omega+5)}$

   Solution : Let

$$\frac{2}{(i\omega+4)(i\omega-3)(i\omega+5)} = \frac{A}{(i\omega+4)} + \frac{B}{(i\omega-3)} + \frac{C}{(i\omega+5)}$$

Then taking the common denominator of the right hand side, we get from the numerator

$$\begin{aligned} 2 &= A(i\omega-3)(i\omega+5) + B(i\omega+4)(i\omega+5) + C(i\omega+4)(i\omega-3) \\ &= (A+B+C)(i\omega)^2 + (-2A+9B+C)i\omega + -15A + 20B - 12C \end{aligned}$$

Thus,

$$\begin{aligned} &A+B+C=0 \\ &-2A+9B+C=0 \\ &-15A+20B-12C=2. \end{aligned}$$

From this we see

$$A = -\frac{10}{3\cdot 21}, \quad B = \frac{1}{21}, \quad C = -A-B = \frac{10}{3\cdot 21} - \frac{1}{21} = \frac{7}{3\cdot 21}$$

Thus, the inverse Fourier transform is

$$\begin{aligned} &A\mathcal{F}^{-1}\Big[\frac{1}{(i\omega+4)}\Big](t) + B\mathcal{F}^{-1}\Big[\frac{1}{(i\omega-3)}\Big](t) + C\mathcal{F}^{-1}\Big[\frac{1}{(i\omega+5)}\Big](t) \\ &= Ae^{-4t}u(t) - Be^{3t}u(-t) + Ce^{-5t}u(t) \\ &= \underline{\frac{10}{3\cdot 21}e^{-4t}u(t) - \frac{1}{21}e^{3t}u(-t) + \frac{7}{3\cdot 21}e^{-5t}u(t)} \end{aligned}$$

(b) $\widehat{g}(\omega) = \frac{1}{-2\omega^2+2i\omega+1}$

Solution :

$$\begin{aligned}
\frac{1}{-2\omega^2+2i\omega+1} &= \frac{1}{2}\frac{1}{(-\omega^2+\omega+\frac{1}{2})} \\
&= \frac{1}{2}\frac{1}{(i\omega+1/2-i/2)(i\omega+1/2+i/2)} \\
&= \frac{1}{2}\frac{1}{(i(\omega+1/2-i/2)(i\omega+1/2+i/2)} \\
&= \frac{-i}{2}\frac{1}{(i\omega+1/2-i/2)} + \frac{i}{2}\frac{1}{(i\omega+1/2+i/2)} \\
&= \frac{-i}{2}\frac{1}{(i(\omega-1/2)+1/2)} + \frac{i}{2}\frac{1}{(i(\omega+1/2)+1/2)}
\end{aligned}$$

Thus, the Fourier inversion is

$$\underline{\frac{-i}{2}e^{-t/2+it/2}u(t) + \frac{i}{2}e^{-t/2-it/2}u(t)}$$

(In the last line, we have used the frequency shifting property in problem 5.)

(c) $\widehat{h}(\omega) = \cos(\omega)\mathrm{sinc}(\omega)$.

(Hint: Express $\cos(\omega)$ in terms of complex exponentials.)

Solution :

$$\cos(\omega)\mathrm{sinc}(\omega) = \frac{1}{2}(e^{i\omega}+e^{-i\omega})\mathrm{sinc}(\omega) = \frac{1}{2}e^{i\omega}\mathrm{sinc}(\omega) + \frac{1}{2}e^{-i\omega}\mathrm{sinc}(\omega)$$

Thus,

$$\begin{aligned}
&\mathcal{F}^{-1}[\cos(\omega)\mathrm{sinc}(\omega)](t) \\
&= \mathcal{F}^{-1}[\frac{1}{2}e^{i\omega}\mathrm{sinc}(\omega)](t) + \mathcal{F}^{-1}[\frac{1}{2}e^{-i\omega}\mathrm{sinc}(\omega)](t) \\
&= \mathcal{F}^{-1}[\frac{1}{2}\mathrm{sinc}(\omega)](t+1) + \mathcal{F}^{-1}[\frac{1}{2}\mathrm{sinc}(\omega)](t-1) \quad \text{(time-shifting)} \\
&= \mathcal{F}^{-1}\left[\frac{1}{4}\mathrm{sinc}(\omega/2)\right](\frac{t+1}{2}) + \mathcal{F}^{-1}\left[\frac{1}{4}\mathrm{sinc}(\omega/2)\right](\frac{t-1}{2}) \quad \text{(scaling)} \\
&= \underline{\frac{1}{4}\mathrm{rect}(\frac{t+1}{2}) + \frac{1}{4}\mathrm{rect}(\frac{t-1}{2})}
\end{aligned}$$

4. Evaluate each integral. (Hint: You may want to recall the definition of Fourier transfrom or Fourier inverse transform, their properties, and basic examples.)

(a) $\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{2}{\omega^2+1}\,d\omega$

Solution : Note that $\frac{1}{\omega^2+1} = \left|\frac{1}{i\omega+1}\right|^2$ and also that $\mathcal{F}[e^{-}tu(t)](\omega) = \frac{1}{i\omega+1}$. Therefore, using Parseval's relation, we get

$$\begin{aligned}
&\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{2}{\omega^2+1}\,d\omega\\
&= 4\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\omega^2+1}\,d\omega\\
&= 4\int_{-\infty}^{\infty}|e^{-t}u(t)|^2 dt\\
&= 4\int_{0}^{\infty}e^{-2t}dt\\
&= \underline{2}.
\end{aligned}$$

(b) $\int_{-\infty}^{\infty}\operatorname{sinc}(4\omega)\,e^{-i4\omega}\,d\omega$

Solution : Ignor this problem.

5. **(Frequency Shifting)**

(a) Show that if $g(t) = e^{i\omega_0 t}f(t)$, then $\widehat{g}(\omega) = \widehat{f}(\omega-\omega_0)$. Also, show that if $\widehat{g}(\omega) = \widehat{f}(\omega-\omega_0)$, then $g(t) = e^{i\omega_0 t}f(t)$. (Hint: Use the definition (the integral expression) of Fourier transform and Fourier inversion.)

Solution For the first part,

$$\begin{aligned}
\mathcal{F}[e^{i\omega_0 t}f(t)](\omega) &= \int_{-\infty}^{\infty}e^{i\omega_0 t}f(t)e^{-it\omega}dt\\
&= \int_{-\infty}^{\infty}f(t)e^{-it(\omega-\omega_0)}dt\\
&= \mathcal{F}[f(t)](\omega-\omega_0).
\end{aligned}$$

Taking Fourier inversion of both the left and right sides, we get the second part.

(b) For a function $h_1(t)$, suppose $\widehat{h}_1(\omega) = \operatorname{sinc}(\frac{\omega}{2}-2)$. Find $h_1(t)$. (Hint: use (a).)

Solution Using the identities in (a),

$$\begin{aligned}
\mathcal{F}^{-1}[\operatorname{sinc}(\omega/2-2)](t) &= \mathcal{F}^{-1}[\operatorname{sinc}(\frac{\omega-4}{2})](t) = e^{i4t}\mathcal{F}^{-1}[\operatorname{sinc}(\omega/2)](t)\\
&= \underline{e^{i4t}\operatorname{rect}(t)}.
\end{aligned}$$

(c) For a function $h_2(t)$, suppose $\widehat{h}_2(\omega) = \operatorname{sinc}(\frac{\omega}{2} - 2) + 2\operatorname{sinc}(\frac{\omega}{2} + 2)$. Find $h_2(t)$. (Hint: use (a) and the linearity of Fourier transform/ inversion.)

Solution

$$\begin{aligned}
&\mathcal{F}^{-1}[\operatorname{sinc}(\frac{\omega}{2} - 2) + 2\operatorname{sinc}(\frac{\omega}{2} + 2)](t) \\
&= \mathcal{F}^{-1}[\operatorname{sinc}(\frac{\omega}{2} - 2)](t) + 2\mathcal{F}^{-1}[\operatorname{sinc}(\frac{\omega}{2} + 2)](t) \\
&= \mathcal{F}^{-1}[\operatorname{sinc}(\frac{\omega - 4}{2})](t) + 2\mathcal{F}^{-1}[\operatorname{sinc}(\frac{\omega + 4}{2})](t) \\
&= e^{i4t}\mathcal{F}^{-1}[\operatorname{sinc}(\frac{\omega}{2})](t) + 2e^{-i4t}\mathcal{F}^{-1}[\operatorname{sinc}(\frac{\omega}{2})](t) \\
&= \underline{e^{i4t}\operatorname{rect}(t) + 2e^{-i4t}\operatorname{rect}(t)}.
\end{aligned}$$

6. (**Differentiation in frequency**)

(a) Prove the following:

$$\text{if } g(t) = tf(t) \text{ then } \widehat{g}(\omega) = i\tfrac{d}{d\omega}\widehat{f}(\omega)$$

(**Hint:** differentiate the definition (I mean, the integral) of $\widehat{f}(\omega)$ with respect to ω: i.e.

$$\frac{d}{d\omega}\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\frac{d}{d\omega}e^{-it\omega}dt. \quad)$$

Solution :

$$\begin{aligned}
i\frac{d}{d\omega}\widehat{f}(\omega) &= i\int_{-\infty}^{\infty} f(t)\frac{d}{d\omega}e^{-it\omega}dt \\
&= i\int_{-\infty}^{\infty} f(t)(-it)e^{-it\omega}dt \\
&= \int_{-\infty}^{\infty} tf(t)e^{-it\omega}dt \\
&= \mathcal{F}[tf(t)](\omega).
\end{aligned}$$

(b) Use (a) to show

$$\text{if } \widehat{g}(\omega) = \tfrac{d}{d\omega}\widehat{f}(\omega), \text{ then } g(t) = -itf(t)$$

Solution : From (a), we know $\mathcal{F}[tf(t)](\omega) = i\frac{d}{d\omega}\widehat{f}(\omega)$. So, multiplying both sides by $-i$, we get $-i\mathcal{F}[tf(t)](\omega) = \frac{d}{d\omega}\widehat{f}(\omega)$. Take inverse Fourier transform to see $-itf(t) = \mathcal{F}^{-1}[\frac{d}{d\omega}\widehat{f}(\omega)](t)$. completing the proof.

(c) Using the frequency differentiation property in part (a), compute the Fourier transform of:

(i) $f(t) = t\,\mathrm{rect}(t)$

Solution :

$$\begin{aligned}\mathcal{F}[t\mathrm{rect}(t)](\omega) &= i\frac{d}{d\omega}\mathcal{F}[\mathrm{rect}(t)](\omega)\\ &= i\frac{d}{d\omega}\mathrm{sinc}(\omega/2)\end{aligned}$$

To compute $\frac{d}{d\omega}\mathrm{sinc}(\omega/2)$, note that $\mathrm{sinc}(\omega/2) = \frac{2}{\omega}\sin(\omega/2)$ for $\omega \neq 0$. So, for $\omega \neq 0$,

$$\begin{aligned}\frac{d}{d\omega}\mathrm{sinc}(\omega/2) &= \frac{d}{d\omega}\frac{2}{\omega}\sin(\omega/2)\\ &= -\frac{2}{\omega^2}\sin(\omega/2) + \frac{2}{\omega}\cos(\omega/2)\frac{1}{2}\\ &= -\frac{2}{\omega^2}\sin(\omega/2) + \frac{1}{\omega}\cos(\omega/2)\\ &= \frac{1}{\omega}\Big[-\frac{2}{\omega}\sin(\omega/2) + \cos(\omega/2)\Big]\end{aligned}$$

Notice that at $\omega = 0$, $\mathrm{sinc}(\omega/2)$ has its maximum and has the horizontal tangent line, so its derivative at $\omega = 0$ is 0. Therefore, we have

$$\mathcal{F}[t\mathrm{rect}(t)](\omega) = i\frac{d}{d\omega}\mathrm{sinc}(\omega/2) = \underline{\begin{cases}\frac{i}{\omega}\Big[-\frac{2}{\omega}\sin(\omega/2) + \cos(\omega/2)\Big] & \text{for } \omega \neq 0,\\ 0 & \text{for } \omega = 0.\end{cases}}$$

(ii) $g(t) = t^2 e^{-3t} u(t)$ (**Hint:** you can apply the frequency differentiation property twice.)

Solution

$$\begin{aligned}\mathcal{F}[t^2 e^{-3t}u(t)](\omega) &= i\frac{d}{d\omega}\mathcal{F}[te^{-3t}u(t)](\omega)\\ &= i\frac{d}{d\omega}i\frac{d}{d\omega}\mathcal{F}[e^{-3t}u(t)](\omega)\\ &= i\frac{d}{d\omega}i\frac{d}{d\omega}\Big[\frac{1}{i\omega+3}\Big]\\ &= i\frac{d}{d\omega}\Big[\frac{1}{(i\omega+3)^2}\Big]\\ &= \underline{\frac{2}{(i\omega+3)^3}}\end{aligned}$$

(d) [Fourier inversion] For a real nonzero constant a, find the function $g(t)$ if

$$\widehat{g}(\omega) = \frac{1}{(i\omega + a)^2}$$

(**Hint:** You can use (b). Can you express $\widehat{g}(\omega)$ as a ω-derivative of certain function?)

Solution

Observe that

$$i\frac{d}{d\omega}\Big[\frac{1}{i\omega + a}\Big] = \frac{1}{(i\omega + a)^2}$$

Also, note that

For the case $a > 0$.

$$\mathcal{F}^{-1}[\frac{1}{i\omega + a}](t) = e^{-at}u(t)$$

For the case $a < 0$ (i.e. $-a > 0$),

$$\begin{aligned}\mathcal{F}^{-1}\Big[\frac{1}{i\omega + a}\Big](t) &= \mathcal{F}^{-1}\Big[-\frac{1}{i(-\omega) + -a}\Big](t) = -e^{-(-a)(-t)}u(-t) \quad \text{(used time reversal)} \\ &= -e^{at}u(-t).\end{aligned}$$

Therefore, by part (a) (or (b)) for the case $a > 0$,

$$\underline{g(t) = te^{-at}u(t),}$$

and for the case $a < 0$,

$$\underline{g(t) = -te^{at}u(-t).}$$

7. **(Optional. Not to be graded.)** It is known that

$$\mathcal{F}\left[e^{-\frac{x^2}{2}}\right](\omega) = \sqrt{2\pi}\, e^{-\frac{\omega^2}{2}}$$

Use this fact to calculate the Fourier transform of $m(x) = x\, e^{-\frac{x^2}{2}}$.

Solution : Notice that

$$m(x) = x\, e^{-\frac{x^2}{2}} = -\frac{d}{dx}e^{-\frac{x^2}{2}}$$

Therefore, from the differentiation property,

$$\begin{aligned}\mathcal{F}[m(x)](\omega) &= -i\omega\mathcal{F}[e^{-\frac{x^2}{2}}](\omega) \\ &= \underline{-i\omega\sqrt{2\pi}\ e^{-\frac{\omega^2}{2}}}\end{aligned}$$

8. **(Optional. Not to be graded.)** **(RLC circuit)** Consider the ODE for RLC circuit:

$$LCy''(t) + RCy'(t) + y(t) = x(t)$$

(a) Let $R = 4$, $L = 3$, $C = 1$ and $\widehat{x}(\omega) = 1$. Find $y(t)$ using Fourier transform method.

Solution The left-hand side is $3y''(t) + 4y'(t) + y(t)$. Thus, the Fourier transform gives

$$-3\omega^2\widehat{y}(\omega) + 4i\omega\widehat{y}(\omega) + \widehat{y}(\omega) = \widehat{x}(\omega).$$

Therefore,

$$\begin{aligned}\widehat{y}(\omega) &= \frac{1}{-3\omega^2 + 4i\omega + 1}\widehat{x}(\omega) \\ &= \frac{1}{-3\omega^2 + 4i\omega + 1} \qquad \text{(since we assumed } \widehat{w}(\omega) = 1.\text{)}\end{aligned}$$

Note that $-3\omega^2 + 4i\omega + 1 = (3i\omega + 1)(i\omega + 1)$ Now, by partial fraction,

$$\begin{aligned}\frac{1}{-3\omega^2 + 4i\omega + 1} &= \frac{1}{(3i\omega + 1)(i\omega + 1)} \\ &= \frac{A}{3i\omega + 1} + \frac{B}{i\omega + 1}.\end{aligned}$$

Here, A and B are determined by

$$\begin{aligned}A(i\omega + 1) + B(3i\omega + 1) &= 1 \\ (A + 3B)i\omega + A + B &= 1\end{aligned}$$

Comparing the real and imaginary parts, we get,

$$A + 3B = 0 \qquad A + B = 1$$

Therefore, $B = -\frac{1}{2}$, $A = \frac{3}{2}$. Thus,

$$\widehat{y}(\omega) = \frac{1}{-3\omega^2 + 4i\omega + 1} = \frac{3}{2(3i\omega + 1)} - \frac{1}{2(i\omega + 1)}$$

Now, for the Fourier inversion $y(t) = \mathcal{F}^{-1}[\widehat{y}(\omega)](t)$,

$$\begin{aligned}
y(t) &= \mathcal{F}^{-1}\Big[\frac{3}{2(3i\omega+1)} - \frac{1}{2(i\omega+1)}\Big](t) \\
&= \frac{3}{2}\mathcal{F}^{-1}\Big[\frac{1}{3i\omega+1}\Big](t) - \frac{1}{2}\mathcal{F}^{-1}\Big[\frac{1}{i\omega+1}\Big](t) \\
&= \frac{1}{2}\mathcal{F}^{-1}\Big[\frac{1}{i\omega+1/3}\Big](t) - \frac{1}{2}\mathcal{F}^{-1}\Big[\frac{1}{i\omega+1}\Big](t) \\
&= \underline{\frac{1}{2}e^{-t/3}u(t) - \frac{1}{2}e^{-t}u(t).}
\end{aligned}$$

(b) Let $R = 2$, $L = 1$, $C = 1$ and $\widehat{x}(\omega) = 1$. Find $y(t)$ using Fourier transform method.

Solution The left-hand side is $y''(t)+2y'(t)+y(t)$. Thus, the Fourier transform gives

$$-\omega^2\widehat{y}(\omega) + 2i\omega\widehat{y}(\omega) + \widehat{y}(\omega) = \widehat{x}(\omega).$$

Therefore,

$$\begin{aligned}
\widehat{y}(\omega) &= \frac{1}{-\omega^2+2i\omega+1}\widehat{x}(\omega) \\
&= \frac{1}{-\omega^2+2i\omega+1} \qquad \text{(since we assumed } \widehat{w}(\omega) = 1.\text{)} \\
&= \frac{1}{(i\omega+1)^2}
\end{aligned}$$

Now use the result of Problem 2 (d), to get

$$y(t) = \underline{te^{-t}u(t).}$$

(c) Let $R = 4$, $L = 3$, $C = 1$ and $x(t) = u(t)e^{-2t}$. Find $y(t)$ using Fourier transform method.

Solution The left-hand side is $3y''(t) + 4y'(t) + y(t)$. Thus, the Fourier transform gives

$$-3\omega^2\widehat{y}(\omega) + 4i\omega\widehat{y}(\omega) + \widehat{y}(\omega) = \widehat{x}(\omega).$$

Therefore,

$$\widehat{y}(\omega) = \frac{1}{-3\omega^2+4i\omega+1}\widehat{x}(\omega)$$

Now, from our class example,

$$\widehat{x}(\omega) = \mathcal{F}[e^{-2t}u(t)](\omega) = \frac{1}{i\omega+2}$$

Therefore,

$$\begin{aligned}\widehat{y}(\omega) &= \frac{1}{-3\omega^2+4i\omega+1}\frac{1}{i\omega+2} \\ &= \frac{1}{(3i\omega+1)(i\omega+1)(i\omega+2)}\end{aligned}$$

(Note that $-3\omega^2+4i\omega+1 = (3i\omega+1)(i\omega+2)$)
Now, by partial fraction,

$$\widehat{y}(\omega) = \frac{A}{3i\omega+1} + \frac{B}{i\omega+1} + \frac{C}{i\omega+2}.$$

Here, A, B and C are determined by

$$A(i\omega+1)(i\omega+2) + B(3i\omega+1)(i\omega+2) + C(3i\omega+1)(i\omega+1) = 1$$

The left hand side is simplified by

$$\begin{aligned}& A(i\omega+1)(i\omega+2) + B(3i\omega+1)(i\omega+2) + C(3i\omega+1)(i\omega+1) \\ &= A(-\omega^2+3i\omega+2) + B(-3\omega^2+7i\omega+2) + C(-3\omega^2+4i\omega+1) \\ &= -(A+3B+3C)\omega^2 + (3A+7B+4C)i\omega + 2A+2B+C\end{aligned}$$

Comparing the last line with 1 (since they should be the same as functions of ω), we have,

$$A+3B+3C=0 \qquad 3A+7B+4C=0 \qquad 2A+2B+C=1$$

Therefore,

$$A = 9/10, \quad B = -1/2, \quad C = 1/5$$

Therefore,

$$\widehat{y}(\omega) = \frac{9}{10}\frac{1}{3i\omega+1} - \frac{1}{2}\frac{1}{i\omega+1} + \frac{1}{5}\frac{1}{i\omega+2}.$$

Therefore,

$$\begin{aligned}y(t) &= \mathcal{F}^{-1}\Big[\widehat{y}(\omega)\Big](t) = \frac{9}{10}\mathcal{F}^{-1}\big[\frac{1}{3i\omega+1}\big](t) - \frac{1}{2}\mathcal{F}^{-1}\big[\frac{1}{i\omega+1}\big](t) + \frac{1}{5}\mathcal{F}^{-1}\big[\frac{1}{i\omega+2}\big](t) \\ &= \frac{9}{30}\mathcal{F}^{-1}\big[\frac{1}{i\omega+1/3}\big](t) - \frac{1}{2}\mathcal{F}^{-1}\big[\frac{1}{i\omega+1}\big](t) + \frac{1}{5}\mathcal{F}^{-1}\big[\frac{1}{i\omega+2}\big](t) \\ &= \frac{3}{10}e^{-t/3}u(t) - \frac{1}{2}e^{-t}u(t) + \frac{1}{5}e^{-2t}u(t) \qquad \text{(by using the standard example } e^{-at}u(t)) \\ &= \underline{u(t)\Big[\frac{3}{10}e^{-t/3} - \frac{1}{2}e^{-t} + \frac{1}{5}e^{-2t}\Big]}\end{aligned}$$

9. **(Optional. Not to be graded.)** What is the inverse Fourier transform of $\widehat{q}(\omega) = e^{+i\omega}\frac{d}{d\omega}\text{sinc}(\omega+1)$?

Answer using properties of Fourier transform / Fourier inversion. (Hint: You may want to use the differentiation in frequency property in one of the previous problems. Alternatively, duality can be useful here.)

Solution : From rearranging, we have

$$e^{-i\omega}\widehat{q}(\omega) = \frac{d}{d\omega}\text{sinc}(\omega+1)$$

Now,

$$\mathcal{F}^{-1}[e^{-i\omega}\widehat{q}(\omega)](t) = \mathcal{F}^{-1}\left[\frac{d}{d\omega}\text{sinc}(\omega+1)\right](t) = -it\mathcal{F}^{-1}\left[\text{sinc}(\omega+1)\right](t)$$

In the last line, we used the ω-differentiation property (Problem 6 (b)). Now, note that using frequency shifting and scaling,

$$\begin{aligned}\mathcal{F}^{-1}\left[\text{sinc}(\omega+1)\right](t) &= e^{-it}\mathcal{F}^{-1}\left[\text{sinc}(\omega)\right](t) = e^{-it}\frac{1}{2}\mathcal{F}^{-1}\left[\text{sinc}(\omega/2)\right](t/2)\\ &= \frac{1}{2}e^{-it}\text{rect}(t/2)\end{aligned}$$

Therefore,

$$\mathcal{F}^{-1}[e^{-i\omega}\widehat{q}(\omega)](t) = -\frac{it}{2}e^{-it}\text{rect}(t/2)$$

Finally from the time-shifting property, we see

$$\underline{\mathcal{F}^{-1}[\widehat{q}(\omega)](t) = -\frac{i(t-1)}{2}e^{-i(t-1)}\text{rect}((t-1)/2)}$$