Math 267, Section 202 : HW 5

Due **Wednesday, February 13th**.

1. Find the Fourier transform of the following functions:

   (a) $f(t) = \begin{cases} \cos(3t) & \text{if } -1 < t < 1, \\ 0 & \text{otherwise.} \end{cases}$ (Hint: express $\cos(3t)$ in terms of complex exponentials.)

   Answer: Note that $\cos(3t) = \frac{1}{2}(e^{3it} + e^{-3it})$. *Thus,*

$$\begin{aligned}
\widehat{f}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \\
&= \int_{-1}^{1} \frac{1}{2}(e^{3it} + e^{-3it})e^{-i\omega t}dt \\
&= \frac{1}{2}\Big\{\int_{-1}^{1} e^{3it}e^{-i\omega t}dt + \int_{-1}^{1} e^{-3it}e^{-i\omega t}dt\Big\} \\
&= \frac{1}{2}\Big\{\int_{-1}^{1} e^{i(3-\omega)t}dt + \int_{-1}^{1} e^{-i(3+\omega)t}dt\Big\}
\end{aligned}$$

   There are three cases (due to the possibility of $3 - \omega = 0$ *or* $3 + \omega = 0$*, which may give zero denominators).*

   Case : $\omega = 3$

$$\begin{aligned}
\widehat{f}(3) &= \frac{1}{2}\Big\{\int_{-1}^{1} dt + \int_{-1}^{1} e^{-i6t}dt\Big\} \\
&= \frac{1}{2}\Big\{2 + \big[\frac{e^{-i6t}}{-i6}\big]_{-1}^{1}\Big\} \\
&= \frac{1}{2}\big(2 + \frac{1}{i6}\big[-e^{-i6} + e^{i6}\big]\big) \\
&= \frac{1}{2}\big(2 + \frac{1}{3}\sin 6\big) \\
&= 1 + \frac{\sin 6}{6}
\end{aligned}$$

   Case : $\omega = -3$

$$\begin{aligned}
\widehat{f}(-3) &= \frac{1}{2}\Big\{\int_{-1}^{1} e^{i4t}dt + \int_{-1}^{1} dt\Big\} \\
&= \frac{1}{2}\Big\{\big[\frac{e^{i6t}}{i6}\big]_{-1}^{1} + 2\Big\} \\
&= \frac{1}{2}\big(\frac{1}{i6}\big[e^{i6} + e^{-i6}\big] + 2\big) \\
&= \frac{1}{2}\big(\frac{1}{3}\sin 6 + 2\big) \\
&= \frac{\sin 6}{6} + 1
\end{aligned}$$

Case: $\omega \neq 3, -3$

$$\begin{aligned}
\widehat{f}(\omega) &= \frac{1}{2}\Big\{\int_{-1}^{1} e^{i(3-\omega)t}dt + \int_{-1}^{1} e^{-i(3+\omega)t}dt\Big\} \\
&= \frac{1}{2}\Big\{\Big[\frac{e^{i(3-\omega)t}}{i(3-\omega)}\Big]_{-1}^{1} - \frac{e^{-i(3+\omega)t}}{i(3+\omega)}\Big]_{-1}^{1}\Big\} \\
&= \frac{1}{2}\Big\{\Big[\frac{e^{i(3-\omega)}}{i(3-\omega)} - \frac{e^{-i(3-\omega)}}{i(3-\omega)}\Big] - \Big[\frac{e^{-i(3+\omega)}}{i(3+\omega)} - \frac{e^{i(3+\omega)}}{i(3+\omega)}\Big]\Big\} \\
&= \frac{1}{2}\Big\{\frac{2}{3-\omega}\sin(3-\omega) + \frac{2}{3+\omega}\sin(3+\omega)\Big\}
\end{aligned}$$

Thus,

$$\widehat{f}(\omega) = \begin{cases} 1 + \frac{\sin 6}{6} & \text{if } \omega = 3 \\ \frac{\sin 6}{6} - 1 & \text{if } \omega = -3 \\ \frac{1}{2}\Big\{\frac{2}{3-\omega}\sin(3-\omega) + \frac{2}{3+\omega}\sin(3+\omega)\Big\} & \text{otherwise.} \end{cases}$$

Remark: *Note that in fact, using the definition of* $\text{sinc}\omega$ *we can rewrite*

$$\widehat{f}(\omega) = \text{sinc}(3-\omega) + \text{sinc}(3+\omega)$$

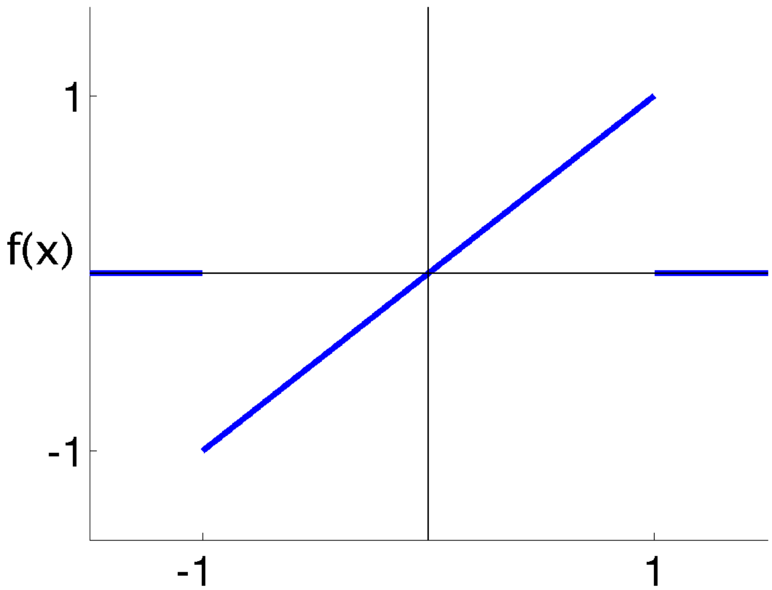
(b) $g(t) = e^{-5|t|}\sin(2t)$. (Hint: express $\sin(2t)$ in terms of complex exponentials.)
Answer: Note that $\sin(2t) = \frac{1}{2i}(e^{2it} - e^{-2it})$*. Thus,*

$$\begin{aligned}
&\widehat{g}(\omega) \\
&= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt \\
&= \int_{-\infty}^{\infty} \frac{1}{2i}(e^{2it} - e^{-2it})e^{-5|t|}e^{-i\omega t}dt \\
&= \int_{-\infty}^{0} \frac{1}{2i}(e^{2it} - e^{-2it})e^{5t}e^{-i\omega t}dt + \int_{0}^{\infty} \frac{1}{2i}(e^{2it} - e^{-2it})e^{-5t}e^{-i\omega t}dt \qquad (|t| = -t \textit{ for } t < 0, \textit{ and } |t| = t \textit{ for } t > 0) \\
&= \int_{-\infty}^{0} \frac{1}{2i}(e^{2it} - e^{-2it})e^{(5-i\omega)t}dt + \int_{0}^{\infty} \frac{1}{2i}(e^{2it} - e^{-2it})e^{-(5+i\omega)t}dt \\
&= \frac{1}{2i}\int_{-\infty}^{0} (e^{\big(5-i(\omega-2)\big)t} - e^{\big(5-i(\omega+2)\big)t}dt \\
&\quad + \frac{1}{2i}\int_{0}^{\infty} (e^{-\big(5+i(\omega+2)\big)t} - e^{-\big(5+i(\omega-2)\big)t}dt \\
&= \frac{1}{2i}\Big[\frac{e^{\big(5-i(\omega-2)\big)t}}{(5-i(\omega-2))} - \frac{e^{\big(5-i(\omega+2)\big)t}}{(5-i(\omega+2))}\Big]_{-\infty}^{0} \\
&\quad + \frac{1}{2i}\Big[\frac{e^{-\big(5+i(\omega+2)\big)t}}{-(5+i(\omega+2))} - \frac{e^{-\big(5+i(\omega-2)\big)t}}{-(5+i(\omega-2))}\Big]_{0}^{\infty}
\end{aligned}$$

Therefore,

$$\widehat{g}(\omega) = \frac{1}{2i}\Big[\frac{1}{(5-i(\omega-2))} - \frac{1}{(5-i(\omega+2))}\Big] + \frac{1}{2i}\Big[\frac{1}{(5+i(\omega+2))} - \frac{1}{(5+i(\omega-2))}\Big]$$

2. By calculating the integral of Fourier transform, find the Fourier transform of $f(x)$:



Hint: You may need to do integration by parts.

Answer

$$\begin{aligned}\mathcal{F}\left[f(x)\right](\omega) &= \int_{-\infty}^{\infty} f(x)e^{-ix\omega}\,dx \\ &= 0 + \int_{-1}^{1} x\,e^{-ix\omega}\,dx + 0\end{aligned}$$

If $\omega = 0$, then $\mathcal{F}\left[f(x)\right](0) = \int_{-1}^{1} x\,e^{0}\,dx = 0$.

If $\omega \neq 0$, then integrate by parts,

$$\begin{aligned}\mathcal{F}\left[f(x)\right](\omega) &= x\frac{e^{-ix\omega}}{-i\omega}\Big|_{x=-1}^{1} - \int_{-1}^{1} 1\cdot\frac{e^{-ix\omega}}{-i\omega}\,dx \\ &= x\frac{e^{-ix\omega}}{-i\omega} - \frac{e^{-ix\omega}}{(-i\omega)^2}\Big|_{x=-1}^{1} \\ &= \frac{e^{-i\omega} - e^{+i\omega}}{-i\omega} + \frac{e^{-i\omega} + e^{+i\omega}}{-\omega^2} \\ &= \frac{2}{\omega}\sin(\omega) - \frac{2}{\omega^2}\cos(\omega)\end{aligned}$$

3. Suppose the Fourier transform of a signal is given by:

$$\widehat{g}(\omega) = \begin{cases} e^{-|\omega|} & \text{for } |\omega| < 4\pi \\ 0 & \text{otherwise} \end{cases}$$

By calculation the integral of Fourier inverse transform, find an explicit formula for $g(t)$. Sketch a plot of $g(t)$.

Answer

$$\begin{aligned} g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{g}(\omega)e^{+i\omega t}\,d\omega \\ &= 0 + \frac{1}{2\pi}\int_{-4\pi}^{0} e^{+\omega}e^{+i\omega t}\,d\omega + \frac{1}{2\pi}\int_{0}^{4\pi} e^{-\omega}e^{+i\omega t}\,d\omega + 0 \\ &= \frac{1-e^{-4\pi(1+it)}}{1+it} + \frac{e^{4\pi(-1+it)}-1}{-1+it} \\ &= \frac{1}{1+it} + \frac{1}{1-it} - e^{-4\pi}\left(\frac{e^{-4\pi it}}{1+it} + \frac{e^{4\pi it}}{1-it}\right) \\ &= \frac{2}{1+t^2} - \frac{e^{-4\pi}}{1+t^2}\left(e^{4\pi it}+e^{-4\pi it}\right) - i\,t\frac{e^{-4\pi}}{1+t^2}\left(e^{4\pi it}-e^{-4\pi it}\right) \\ &= \frac{2}{1+t^2} - \frac{e^{-4\pi}}{1+t^2}\left(2\cos(4\pi t) - 2\sin(4\pi t)\right) \end{aligned}$$

4. Use basic examples and properties of Fourier transform to find the Fourier transform of each signal. You are expected to answer these questions without calculating integrals.

   (a) $g(t) = \begin{cases} 1 & \text{for } -1 < t < 0 \\ -1 & \text{for } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$

   (b) $f(t) = e^{3t}$ for $t < \pi$, and $f(t) = 0$ for $t \geq \pi$.

   (c) $h(t) = e^{3t}$ for $0 < t < \pi$, and $h(t) = 0$ otherwise.

Answer

(a) Recognize $g(t)$ is a sum of rectangles, then use time-shifting property.

$$\begin{aligned} g(t) &= \text{rect}(t+\frac{1}{2}) - \text{rect}(t-\frac{1}{2}) \\ \widehat{g}(\omega) &= e^{i\omega\,\frac{1}{2}}\text{sinc}(\frac{\omega}{2}) - e^{-i\omega\,\frac{1}{2}}\text{sinc}(\frac{\omega}{2}) \end{aligned}$$

(b) $e^{-3t}u(t)$, a basic example, nonzero for $t > 0$
$e^{-3(-t)}u(-t)$, reflection, nonzero for $t < 0$
$e^{-3(-(t-\pi))}u(-(t-\pi))$, shift to the right, nonzero for $t < \pi$

$$f(t) = e^{3\pi}\left(e^{-3(-(t-\pi))}u(-(t-\pi))\right)$$

In brackets is a shift and scaled basic example, with $\phi_0 = -\pi$ and $\lambda_0 = -1$.

$$\widehat{f}(\omega) = e^{3\pi}\left(e^{-\pi\omega}\frac{1}{|-1|}\mathcal{F}\left[e^{-3t}u(t)\right]\left(\frac{\omega}{-1}\right)\right) = e^{3\pi}\,e^{-\pi\omega}\frac{1}{3+i(-\omega)}$$

(c) We'll take a slightly different approach. First,

$$h(t) = e^{3t}\left(u(\pi - t) - u(-t)\right),$$

since $u(\pi - t) = 1$ for $\pi > t$ and $u(-t) = 1$ for $0 > t$. Now expand, rewrite so the variable only shows up as it does in the argument of the Heaviside functions:

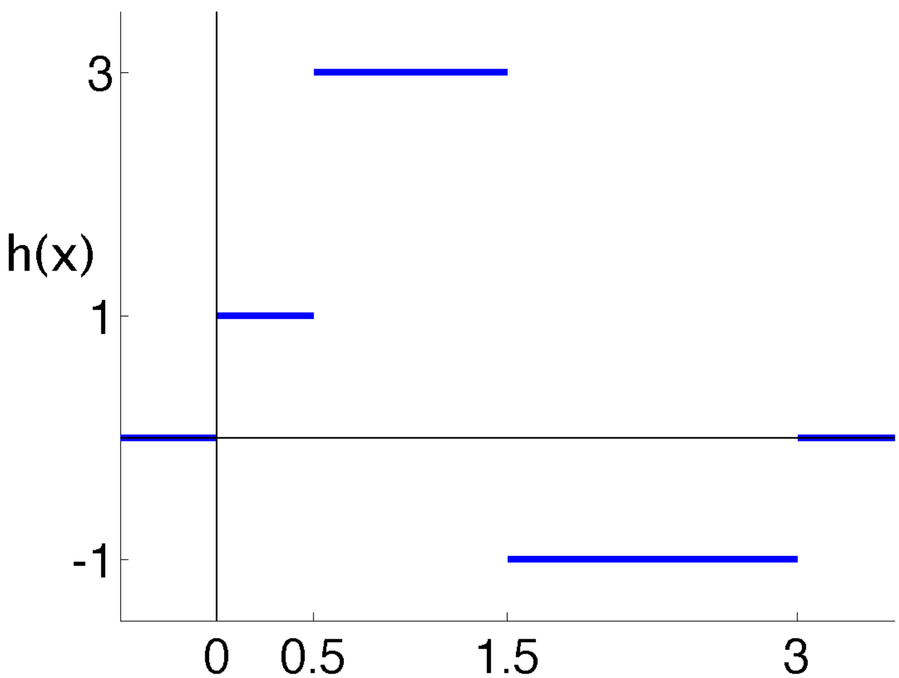
$$h(t) = e^{3t}u(\pi - t) - e^{3t}u(-t) = e^{3\pi}e^{-3(\pi-t)}u(\pi - t) - e^{-3(-t)}u(-t)$$

$$\widehat{h}(\omega) = e^{3\pi}\, e^{-\pi\omega}\frac{1}{3 + i(-\omega)} - \frac{1}{3 + i(-\omega)}$$

5. Consider $h(x)$ given by the plot below.

(a) Write $h(x)$ as a sum of $\mathrm{rect}\left(\lambda_0(x - \phi_0)\right)$, for various λ_0 and ϕ_0.
Note: there is more than one way to do this.

(b) Use properties of Fourier transform to compute $\widehat{h}(\omega)$.



Answer

(a) The easiest answer is,

$$h(x) = \mathrm{rect}\left(2(x - \frac{1}{4})\right) + 3\,\mathrm{rect}\left(x - 1\right) - \mathrm{rect}\left(\frac{2}{3}(x - \frac{9}{4})\right)$$

(b) Don't forget the factor of $\frac{1}{2}$ from $\mathcal{F}\left[\mathrm{rect}(x)\right](\omega) = \mathrm{sinc}\left(\frac{\omega}{2}\right)$.

$$\widehat{h}(\omega) = e^{-i\frac{\omega}{4}}\frac{1}{2}\mathrm{sinc}\left(\frac{\omega}{4}\right) + 3\,e^{-i\omega}\mathrm{sinc}\left(\frac{\omega}{2}\right) - e^{-i\frac{9\omega}{4}}\frac{3}{2}\mathrm{sinc}\left(3\omega\right)$$

6. **(This exercise is optional and NOT to be graded. It is intended to give you a better understanding of the relation between Fourier series and Fourier transform.)** For $L > 0$, let $\omega_k = k\frac{\pi}{L}$, for $k = 0, \pm1, \pm2, \pm3, \pm4, \cdots$, and let $\Delta\omega = \frac{\pi}{L}$.

   (a) Show that the Fourier series of a $2L$-periodic function $f(t)$ can be written as

$$f(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \widetilde{f}(\omega_k) e^{i\omega_k t} \Delta\omega,$$

   where $i = \sqrt{-1}$ and

$$\widetilde{f}(\omega_k) = \int_{-L}^{L} f(t) e^{-i\omega_k t} dt.$$

   Answer: Note that the Fourier series of a $2L$-periodic function $f(t)$ is

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\frac{\pi}{L}t}$$

   where

$$c_k = \frac{1}{2L} \int_{-L}^{L} f(t) e^{-ik\frac{\pi}{L}t} dt.$$

   First, using $\omega_k = k\frac{\pi}{L}$, we can write

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k t}$$

   Now,

$$\begin{aligned} c_k &= \frac{1}{2L} \int_{-L}^{L} f(t) e^{-i\omega_k t} dt \\ &= \frac{1}{2\pi} \frac{\pi}{L} \int_{-L}^{L} f(t) e^{-i\omega_k t} dt \\ &= \frac{1}{2\pi} \Delta\omega \int_{-L}^{L} f(t) e^{-i\omega_k t} dt \qquad \textit{(using the definition of } \Delta\omega\textit{)} \\ &= \frac{1}{2\pi} \left\{ \int_{-L}^{L} f(t) e^{-i\omega_k t} dt \right\} \Delta\omega \\ &= \frac{1}{2\pi} \widetilde{f}(\omega_k) \Delta\omega \qquad \textit{(using the definition of } \widetilde{f}(\omega_k)\textit{)} \end{aligned}$$

   Back to the Fourier series, we get

$$f(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \widetilde{f}(\omega_k) e^{i\omega_k t} \Delta\omega$$

   as desired.

   (b) Use the notation of (a). Assume $L > 1$. Let $g(t)$ be the $2L$**-periodic** function given by

$$g(t) = \begin{cases} 1 & \text{if } -1/2 < t < 1/2, \\ 0 & \text{if } -L < t \le -1/2, \\ 0 & \text{if } 1/2 \le t < L. \end{cases}$$

Compute $\widetilde{g}(\omega_k) = \int_{-L}^{L} g(t)e^{-i\omega_k t}dt$, for all integer k.
Answer

$$\widetilde{g}(\omega_k) = \int_{-L}^{L} g(t)e^{-i\omega_k t}dt$$
$$= \int_{-1/2}^{1/2} e^{-i\omega_k t}dt.$$

Now, two cases:
Case: $\omega_k = k\pi/L = 0$**, i.e.** $k = 0$*:*

$$\widetilde{g}(\omega_k) = \int_{-1/2}^{1/2} dt = 1.$$

Case: $\omega_k = k\pi/L \neq 0$**, i.e.** $k \neq 0$*:*

$$\begin{aligned}
\widetilde{g}(\omega_k) &= \int_{-1/2}^{1/2} e^{-i\omega_k t}dt \\
&= \left[\frac{e^{-i\omega_k t}}{-i\omega_k}\right]_{-1/2}^{1/2} \\
&= \frac{e^{-i\omega_k/2}}{-i\omega_k} - \frac{e^{i\omega_k/2}}{-i\omega_k} \\
&= \frac{1}{i\omega_k}\left[e^{i\omega_k/2} - e^{-i\omega_k/2}\right] \qquad \textit{(rearranged)} \\
&= \frac{1}{i\omega_k} 2i\sin\left(\frac{\omega_k}{2}\right) \qquad \textit{(notice that } 2i\sin\theta = e^{i\theta} - e^{-i\theta} \textit{)} \\
&= \frac{2}{\omega_k}\sin(\frac{\omega_k}{2}).
\end{aligned}$$

Thus,

$$\underline{\widetilde{g}(\omega_k) = \begin{cases} 1 & \textit{if } \omega_k = 0 \\ \frac{2}{\omega_k}\sin(\frac{\omega_k}{2}) & \textit{otherwise.} \end{cases}}$$

(Notice that the last function is exactly $\text{sinc}(\omega_k/2)$*.)*

(c) For the function $g(t)$ in (b), do the following.

i. Let $L = 1$. Sketch the graph of $\widetilde{g}(\omega_k)$ for $-6\pi < \omega_k < 6\pi$. For the values ω_k, use the ω-axis (i.e. the horizontal axis with ω variable)

ii. Let $L = 2$. Sketch the graph of $\widetilde{g}(\omega_k)$ for $-6\pi < \omega_k < 6\pi$. Use the ω-axis for the values ω_k.

iii. Let $L = 4$. Sketch the graph of $\widetilde{g}(\omega_k)$ for $-6\pi < \omega_k < 6\pi$. Use the ω-axis for the values ω_k.

AnswerWe skip this part.

7. **(This exercise will NOT be graded. This is an optional exercise for how to use properties of Fourier transform)** In this problem, it is useful to recall that the Fourier transform of the function

$$\text{rect(t)} = \begin{cases} 1 & \text{if } -\frac{1}{2} < t < \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

is the function $\text{sinc}(\frac{\omega}{2})$, i.e. $\underline{\widehat{\text{rect}}(\omega) = \text{sinc}(\frac{\omega}{2})}$ where the function sinc is defined as

$$\text{sinc}(\omega) = \begin{cases} 1 & \text{if } \omega = 0, \\ \frac{\sin(\omega)}{\omega} & \text{otherwise.} \end{cases}$$

(See the online notes "Fourier Transform" page 2, Example 2.)

(a) Find the Fourier transform of

$$f(t) = \begin{cases} e^{-t} & \text{if } 0 < t < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(Hint: just directly compute the integral for the Fourier transform.)
Answer

$$\begin{aligned} \widehat{f}(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\ &= \int_0^1 e^{-t} e^{-i\omega t} dt \\ &= \int_0^1 e^{-(1+i\omega)t} dt \\ &= \left[\frac{e^{-(1+i\omega)t}}{-(1+i\omega)} \right]_0^1 \\ &= \underline{\frac{1}{1+i\omega}\Big[- e^{-(1+i\omega)} + 1 \Big]} \end{aligned}$$

(b) Without calculating integrals directly, **but only using** the properties of the Fourier transform and the result of (a),

i. find $\widehat{g}_1(\omega)$ where $g_1(t) = \begin{cases} e^{-t+4} & \text{if } 4 < t < 5, \\ 0 & \text{otherwise.} \end{cases}$

Answer: Note that $g_1(t) = f(t-4)$ *since* $f(t-4) = \begin{cases} e^{-(t-4)} & \textit{if } 0 < t-4 < 1, \\ 0 & \textit{otherwise} \end{cases} = \begin{cases} e^{-t+4} & \textit{if } 4 < t < 5, \\ 0 & \textit{otherwise.} \end{cases}$ *Thus, by the time-shift property of Fourier series*

$$\begin{aligned} \widehat{g}_1(\omega) &= e^{-i4\omega} \widehat{f}(\omega) \\ &= \underline{e^{-i4\omega} \frac{1}{1+i\omega}\Big[- e^{-(1+i\omega)} + 1 \Big]} \end{aligned}$$

ii. find $\widehat{g}_2(\omega)$ where $g_2(t) = \begin{cases} e^{-t} & \text{if } 4 < t < 5, \\ 0 & \text{otherwise.} \end{cases}$

Answer: Note that $g_2(t) = e^{-4} g_1(t)$. *Thus, (by linearity of Fourier transform),* $\widehat{g}_2(\omega) = e^{-4}\widehat{g}_1(\omega)$, *therefore,*

$$\widehat{g}_2(\omega) = \underline{e^{-4-i4\omega} \frac{1}{1+i\omega}\Big[- e^{-(1+i\omega)} + 1 \Big]}$$

(Hint: consider the time-shifting property of the Fourier transform.)

(c) Use above (including the results about rect(t)) and **linearity and time-shifting property** for Fourier transform, to find Fourier transform of the following functions, i.e. **without** calculating integrals directly.

i. $h_1(t) = \begin{cases} 1 & \text{if } 0 < t < 1, \\ -2 & \text{if } 3 < t < 4, \\ e^{-t+4} & \text{if } 4 < t < 5, \\ 0 & \text{otherwise.} \end{cases}$

*Answer*Note that $h_1(t) = \mathrm{rect(t-1/2) - 2rect(t-7/2) + g_1(t)}$.
Let us use the notation $\mathcal{F}[f(t)](\omega)$ *(or* $\mathcal{F}[f](\omega)$ *) for the Fourier transform of* $f(t)$*. So, by notation* $\widehat{f}(\omega) = \mathcal{F}[f(t)])(\omega)$*. Thus,*

$$\mathcal{F}[h_1(t)](\omega) = \mathcal{F}[\mathrm{rect(t-1/2)}](\omega) - 2\mathcal{F}[\mathrm{rect(t-7/2)}](\omega) + \mathcal{F}[\mathrm{g_1(t)}](\omega)$$

by the linearity of the Fourier transform.
Now, use the time-shift property of Fourier transform, so see

$$\begin{aligned} \mathcal{F}[h_1(t)](\omega) &= e^{-i\frac{1}{2}\omega}\mathcal{F}[\mathrm{rect(t)}](\omega) - 2\mathrm{e}^{-\mathrm{i}\frac{7}{2}\omega}\mathcal{F}[\mathrm{rect(t)}](\omega) + \mathcal{F}[\mathrm{g_1(t)}](\omega) \\ &= e^{-i\frac{1}{2}\omega}\mathrm{sinc}(\omega/2) - 2\mathrm{e}^{-\mathrm{i}\frac{7}{2}\omega}\mathrm{sinc}(\omega/2) + \mathrm{e}^{-\mathrm{i}4\omega}\frac{1}{1+\mathrm{i}\omega}\Big[-\mathrm{e}^{-(1+\mathrm{i}\omega)} + 1\Big] \\ &= \underline{\Big(e^{-i\frac{1}{2}\omega} - 2e^{-i\frac{7}{2}\omega}\Big)\mathrm{sinc}(\omega/2) + \mathrm{e}^{-\mathrm{i}4\omega}\frac{1}{1+\mathrm{i}\omega}\Big[-\mathrm{e}^{-(1+\mathrm{i}\omega)} + 1\Big]} \end{aligned}$$

ii. $h_2(t) = \begin{cases} -1 & \text{if } -1 < t < 0, \\ 3 & \text{if } 1 < t < 2, \\ e^{-t} & \text{if } 4 < t < 5, \\ 0 & \text{otherwise.} \end{cases}$

*Answer*Note that $h_2(t) = \mathrm{-rect(t+1/2) + 3rect(t-3/2) + g_2(t)}$.
Let us use the notation $\mathcal{F}[f(t)](\omega)$ *(or* $\mathcal{F}[f](\omega)$ *) for the Fourier transform of* $f(t)$*. So, by notation* $\widehat{f}(\omega) = \mathcal{F}[f(t)])(\omega)$*. Thus,*

$$\mathcal{F}[h_2(t)](\omega) = -\mathcal{F}[\mathrm{rect(t+1/2)}](\omega) + 3\mathcal{F}[\mathrm{rect(t-3/2)}](\omega) + \mathcal{F}[\mathrm{g_2(t)}](\omega)$$

by the linearity of the Fourier transform.
Now, use the time-shift property of Fourier transform, so see

$$\begin{aligned} \mathcal{F}[h_1(t)](\omega) &= -e^{+i\frac{1}{2}\omega}\mathcal{F}[\mathrm{rect(t)}](\omega) + 3\mathrm{e}^{-\mathrm{i}\frac{3}{2}\omega}\mathcal{F}[\mathrm{rect(t)}](\omega) + \mathcal{F}[\mathrm{g_2(t)}](\omega) \\ &= -e^{+i\frac{1}{2}\omega}\mathrm{sinc}(\omega/2) + 3\mathrm{e}^{-\mathrm{i}\frac{3}{2}\omega}\mathrm{sinc}(\omega/2) + \mathrm{e}^{-4-\mathrm{i}4\omega}\frac{1}{1+\mathrm{i}\omega}\Big[-\mathrm{e}^{-(1+\mathrm{i}\omega)} + 1\Big] \\ &= \underline{\Big(-e^{+i\frac{1}{2}\omega} + 3e^{-i\frac{3}{2}\omega}\Big)\mathrm{sinc}(\omega/2) + \mathrm{e}^{-4-\mathrm{i}4\omega}\frac{1}{1+\mathrm{i}\omega}\Big[-\mathrm{e}^{-(1+\mathrm{i}\omega)} + 1\Big]} \end{aligned}$$

8. **(This problem will NOT be graded.)** Sketch the graph of the following functions. Then find their Fourier transforms using appropriate properties of Fourier transform together with your answer to Problem 7(a), **without** directly calculating the integral for the Fourier transform.

(a) $f(t) = \begin{cases} e^{t} & \text{if } -1 < t < 0, \\ 0 & \text{otherwise.} \end{cases}$

*Answer**Rename the function in 1 (a) as* $f_1(t)$*. Then, the function* $f(t)$ *(in the current problem)*

$$f(t) = f_1(-t).$$

Then,

$$\widehat{f}(\omega) = \widehat{f}_1(-\omega) \text{ (using 2 (a))}$$

Now, using 1 (a)

$$\widehat{f}_1(-\omega) = \frac{1}{1+i(-\omega)}\Big[-e^{-(1+i(-\omega))}+1\Big]$$

Therefore,

$$\begin{aligned}\widehat{f}(\omega) &= \frac{1}{1+i(-\omega)}\Big[-e^{-(1+i(-\omega))}+1\Big]\\ &= \underline{\frac{1}{1-i\omega}\Big[-e^{-(1-i\omega)}+1\Big]}\end{aligned}$$

(b)

$$g(t) = \begin{cases} e^t, & -1<t<0\\ e^{-t}, & 0<t<1\\ 0, & \text{otherwise}\end{cases}$$

$$h(t) = \begin{cases} -e^t, & -1<t<0\\ e^{-t}, & 0<t<1\\ 0, & \text{otherwise}\end{cases}$$

$$s(t) = \begin{cases} e^{-t-1}, & -1<t<0\\ e^{-t}, & 0<t<1\\ 0, & \text{otherwise}\end{cases}$$

AnswerLet (for notation)

$$f_1(t) = \begin{cases} e^{-t} & \text{if } 0<t<1,\\ 0 & \text{otherwise.}\end{cases}$$

and let

$$f_2(t) = \begin{cases} e^{t} & \text{if } -1<t<0,\\ 0 & \text{otherwise.}\end{cases}$$

Note by (a) and 1 (a), we see that

$$\widehat{f}_1(\omega) = \frac{1}{1+i\omega}\Big[-e^{-(1+i\omega)}+1\Big]$$
$$\widehat{f}_2(\omega) = \frac{1}{1-i\omega}\Big[-e^{-(1-i\omega)}+1\Big]$$

Then, using this notation, we see that $g(t) = f_2(t) + f_1(t)$, $h(t) = -f_2(t) + f_1(t)$, *and* $s(t) = f_1(t+1) + f_1(t)$.
Thus, using the linearity of Fourier transform and time-shift property,

$$\begin{aligned}\widehat{g}(\omega) &= \widehat{f}_2(\omega) + \widehat{f}_1(\omega)\\ \widehat{h}(\omega) &= -\widehat{f}_2(\omega) + \widehat{f}_1(\omega)\\ \widehat{s}(\omega) &= e^{i\omega}\widehat{f}_1(\omega) + \widehat{f}_1(\omega) \quad \text{(used time-shift by } -1)\\ &= (e^{i\omega}+1)\widehat{f}_1(\omega)\end{aligned}$$

Therefore, we get

$$
\begin{aligned}
\widehat{g}(\omega) &= \frac{1}{1-i\omega}\Big[-e^{-(1-i\omega)}+1\Big] + \frac{1}{1+i\omega}\Big[-e^{-(1+i\omega)}+1\Big] \\
\widehat{h}(\omega) &= -\frac{1}{1-i\omega}\Big[-e^{-(1-i\omega)}+1\Big] + \frac{1}{1+i\omega}\Big[-e^{-(1+i\omega)}+1\Big] \\
\widehat{s}(\omega) &= (e^{i\omega}+1)\frac{1}{1+i\omega}\Big[-e^{-(1+i\omega)}+1\Big]
\end{aligned}
$$