Math 267, Section 202 : HW 5

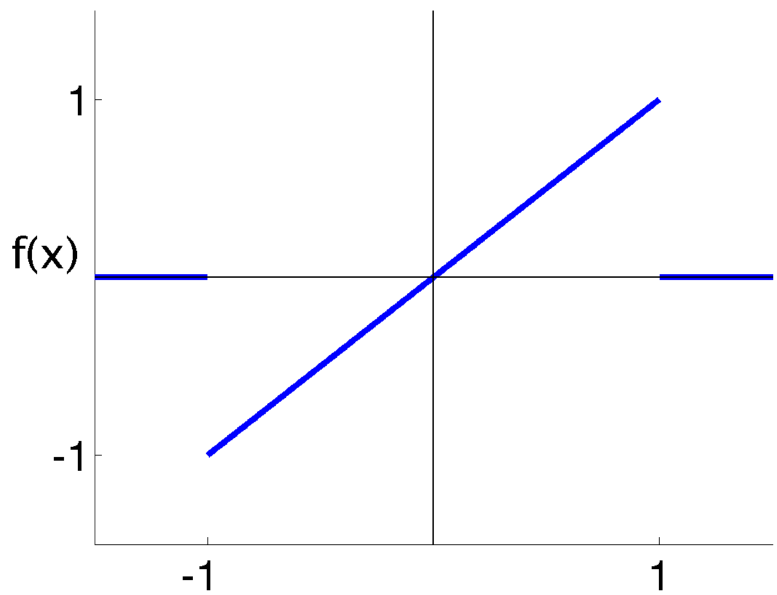
Due **Wednesday, February 13th**.

1. Find the Fourier transform of the following functions:

   (a) $f(t) = \begin{cases} \cos(3t) & \text{if } -1 < t < 1, \\ 0 & \text{otherwise.} \end{cases}$ (Hint: express $\cos(3t)$ in terms of complex exponentials.)

   (b) $g(t) = e^{-5|t|}\sin(2t)$. (Hint: express $\sin(2t)$ in terms of complex exponentials.)

2. By calculating the integral of Fourier transform, find the Fourier transform of $f(x)$:



   Hint: You may need to do integration by parts.

3. Suppose the Fourier transform of a signal is given by:

$$\widehat{g}(\omega) = \begin{cases} e^{-|\omega|} & \text{for } |\omega| < 4\pi \\ 0 & \text{otherwise} \end{cases}$$

   By calculation the integral of Fourier inverse transform, find an explicit formula for $g(t)$. Sketch a plot of $g(t)$.

4. Use basic examples and properties of Fourier transform to find the Fourier transform of each signal. You are expected to answer these questions without calculating integrals.

   (a) $g(t) = \begin{cases} 1 & \text{for } -1 < t < 0 \\ -1 & \text{for } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$

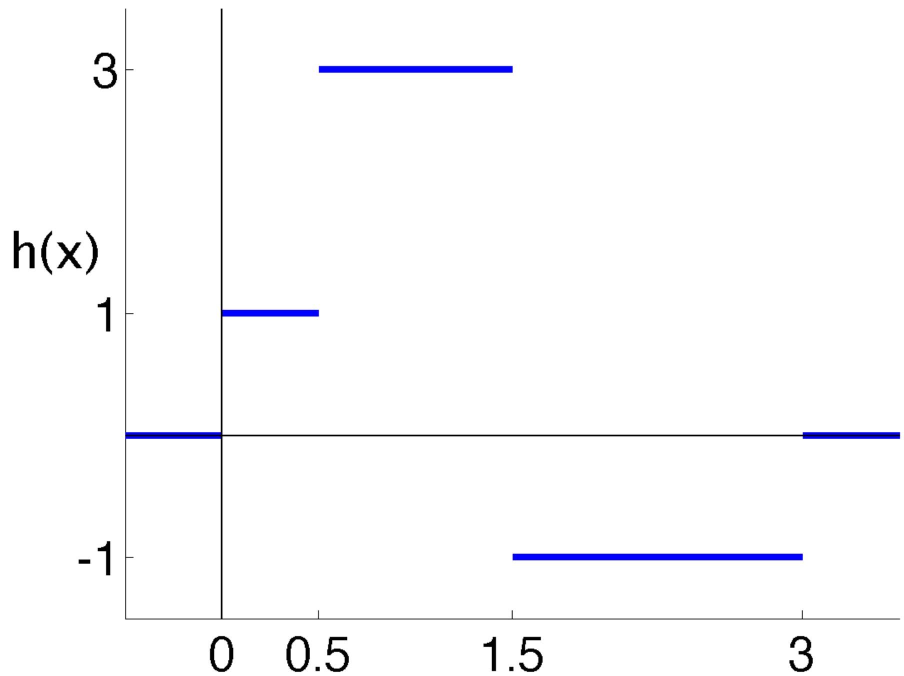
   (b) $f(t) = e^{3t}$ for $t < \pi$, and $f(t) = 0$ for $t \geq \pi$.

   (c) $h(t) = e^{3t}$ for $0 < t < \pi$, and $h(t) = 0$ otherwise.

5. Consider $h(x)$ given by the plot below.

   (a) Write $h(x)$ as a sum of $\text{rect}\,(\lambda_0(x - \phi_0))$, for various λ_0 and ϕ_0.
   Note: there is more than one way to do this.

   (b) Use properties of Fourier transform to compute $\widehat{h}(\omega)$.



6. **(This exercise is optional and NOT to be graded. It is intended to give you a better understanding of the relation between Fourier series and Fourier transform.)** For $L > 0$, let $\omega_k = k\frac{\pi}{L}$, for $k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \cdots$, and let $\Delta\omega = \frac{\pi}{L}$.

   (a) Show that the Fourier series of a $2L$-periodic function $f(t)$ can be written as

$$f(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \widetilde{f}(\omega_k) e^{i\omega_k t} \Delta\omega,$$

   where $i = \sqrt{-1}$ and

$$\widetilde{f}(\omega_k) = \int_{-L}^{L} f(t) e^{-i\omega_k t} dt.$$

   (b) Use the notation of (a). Assume $L > 1$. Let $g(t)$ be the $2L$**-periodic** function given by

$$g(t) = \begin{cases} 1 & \text{if } -1/2 < t < 1/2, \\ 0 & \text{if } -L < t \le -1/2, \\ 0 & \text{if } 1/2 \le t < L. \end{cases}$$

   Compute $\widetilde{g}(\omega_k) = \int_{-L}^{L} g(t) e^{-i\omega_k t} dt$, for all integer k.

(c) For the function $g(t)$ in (b), do the following.

i. Let $L = 1$. Sketch the graph of $\widetilde{g}(\omega_k)$ for $-6\pi < \omega_k < 6\pi$. For the values ω_k, use the ω-axis (i.e. the horizontal axis with ω variable)

ii. Let $L = 2$. Sketch the graph of $\widetilde{g}(\omega_k)$ for $-6\pi < \omega_k < 6\pi$. Use the ω-axis for the values ω_k.

iii. Let $L = 4$. Sketch the graph of $\widetilde{g}(\omega_k)$ for $-6\pi < \omega_k < 6\pi$. Use the ω-axis for the values ω_k.

7. **(This exercise will NOT be graded. This is an optional exercise for how to use properties of Fourier transform)** In this problem, it is useful to recall that the Fourier transform of the function

$$\text{rect(t)} = \begin{cases} 1 & \text{if } -\frac{1}{2} < t < \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

is the function $\text{sinc}(\frac{\omega}{2})$, i.e. $\underline{\widehat{\text{rect}}(\omega) = \text{sinc}(\frac{\omega}{2})}$ where the function sinc is defined as

$$\text{sinc}(\omega) = \begin{cases} 1 & \text{if } \omega = 0, \\ \frac{\sin(\omega)}{\omega} & \text{otherwise.} \end{cases}$$

(See the online notes "Fourier Transform" page 2, Example 2.)

(a) Find the Fourier transform of

$$f(t) = \begin{cases} e^{-t} & \text{if } 0 < t < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(Hint: just directly compute the integral for the Fourier transform.)

(b) Without calculating integrals directly, **but only using** the properties of the Fourier transform and the result of (a),

i. find $\widehat{g}_1(\omega)$ where $g_1(t) = \begin{cases} e^{-t+4} & \text{if } 4 < t < 5, \\ 0 & \text{otherwise.} \end{cases}$

ii. find $\widehat{g}_2(\omega)$ where $g_2(t) = \begin{cases} e^{-t} & \text{if } 4 < t < 5, \\ 0 & \text{otherwise.} \end{cases}$

(Hint: consider the time-shifting property of the Fourier transform.)

(c) Use above (including the results about rect(t)) and **linearity and time-shifting property** for Fourier transform, to find Fourier transform of the following functions, i.e. **without** calculating integrals directly.

i. $h_1(t) = \begin{cases} 1 & \text{if } 0 < t < 1, \\ -2 & \text{if } 3 < t < 4, \\ e^{-t+4} & \text{if } 4 < t < 5, \\ 0 & \text{otherwise.} \end{cases}$

ii. $h_2(t) = \begin{cases} -1 & \text{if } -1 < t < 0, \\ 3 & \text{if } 1 < t < 2, \\ e^{-t} & \text{if } 4 < t < 5, \\ 0 & \text{otherwise.} \end{cases}$

8. **(This problem will NOT be graded.)** Sketch the graph of the following functions. Then find their Fourier transforms using appropriate properties of Fourier transform together with your answer to Problem 7(a), **without** directly calculating the integral for the Fourier transform.

(a) $f(t) = \begin{cases} e^t & \text{if } -1 < t < 0, \\ 0 & \text{otherwise.} \end{cases}$

(b)

$$\begin{aligned} g(t) &= \begin{cases} e^t, & -1 < t < 0 \\ e^{-t}, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases} \\ h(t) &= \begin{cases} -e^t, & -1 < t < 0 \\ e^{-t}, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases} \\ s(t) &= \begin{cases} e^{-t-1}, & -1 < t < 0 \\ e^{-t}, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$