Math 267 : HW 2

All questions are due **Monday January 21st**

1. Consider the wave equation with boundary conditions (here, we do not specify the initial conditions):

$$\begin{cases} u_{tt} = u_{xx} \ , & 0 < x < 1, \, t > 0, \\ u(0,t) = 0 = u_x(1,t), & t > 0. \end{cases}$$

   (Notice the derivative in one of the boundary conditions!) Find all nontrivial (i.e. not identically zero) solutions of type

$$u(x,t) = X(x)T(t).$$

   Here, you have to solve a relevant eigenvalue problem.

2. (a) Find all nontrivial (i.e. not identically zero) solutions of type

$$u(x,t) = X(x)T(t),$$

   to the following wave equation and given boundary conditions

$$\begin{cases} u_{tt} = 9u_{xx} \ , & 0 < x < 2, \, t > 0, \\ u(0,t) = 0 = u(2,t), & t > 0. \end{cases}$$

   Here, you have to solve a relevant eigenvalue problem.

   (b) Use (a) to find a general solution solving the same equation and the boundary conditions:

   (c) Use (a) and (b) to find the solution to the same equation and the boundary condition, and moreover the following initial condition:

$$u(x,0) = \sin(\pi x) + 2\sin(5\pi x), \qquad u_t(x,0) = 1.$$

3. Assume that,

$$\sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi}{2}x\right) = \begin{cases} -1 & \text{for } 0 < x < 1 \\ +1 & \text{for } 1 < x < 2 \end{cases}$$

   Find the coefficients c_k. Evaluate c_k when k is even, and also when k is odd.

4. Assume that,

$$e^{-x} = \sum_{k=1}^{\infty} c_k \sin(kx) \, ,$$

   for $0 < x < \pi$. Find the coefficients c_k.

   (Hint: You may need to compute some integrals, using integration by parts twice.)

5. Find a solution $y(t)$ of the form $y(t) = \sum_{k=1}^{\infty} c_k \sin(\frac{k\pi}{2}t)$ on the interval $0 < t < 2$, solving the following boundary value problem:

$$\begin{cases} y''(t) + y(t) = f(t) & \text{for } 0 < t < 2 \ , \\ y(0) = 0, \quad y(2) = 0, \end{cases}$$

   where

$$f(t) = \begin{cases} 0 & \text{for } 0 < t < 1, \\ 1 & \text{for } 1 < t < 2. \end{cases}$$

   In other words, determine the constants c_k for each $k = 1, 2, 3, \cdots$, so that $y(t)$ solves the boundary value problem. In particular, evaluate c_1 and c_2.