#1 solution:

Set $u(x,t) = X(x)T(t)$ and plug into the equation we get two separated ODE:

$X'' - \sigma X = 0$ and $T'' - \sigma T = 0$.

If $\sigma = 0$, then $u(x,t) = (c_1 + c_2 x)(c_3 + c_4 t)$,

If $\sigma \neq 0$, then $u(x,t) = \left(c_1 e^{\sqrt{\sigma}x} + c_2 e^{-\sqrt{\sigma}x}\right)\left(c_3 e^{\sqrt{\sigma}t} + c_4 e^{-\sqrt{\sigma}t}\right)$.

Now use the boundary conditions:

Case 1: $\sigma = 0$:

$u(0,t) = X(0)T(t) = 0$, then $X(0) = 0$, hence $c_1 = 0$,

$u_x(1,t) = X'(1)T(t) = 0$, then $X'(1) = 0$, hence $c_2 = 0$.

Hence the solution is trivial.

Case 2: $\sigma \neq 0$:

$u(0,t) = X(0)T(t) = 0$, then $X(0) = 0$, then $c_1 + c_2 = 0$,

$u_x(1,t) = X'(1)T(t) = 0$, then X'(1)=0, then $c_1\sqrt{\sigma}e^{\sqrt{\sigma}} - c_2\sqrt{\sigma}e^{-\sqrt{\sigma}} = 0$.

Hence combining the above two conditions we get $e^{2\sqrt{\sigma}} = -1$, and $\sqrt{\sigma} = \frac{(2k+1)\pi}{2}i$, $\sigma = -\frac{(2k+1)^2\pi^2}{4}$.

The general solution is

$u(x,t) = c_1\left(e^{i\frac{2k+1}{2}\pi x} - e^{-i\frac{2k+1}{2}\pi x}\right)\left(c_3 e^{i\frac{2k+1}{2}\pi t} + c_4 e^{-i\frac{2k+1}{2}\pi t}\right)$

$= \sin\left(\frac{2k+1}{2}\pi x\right)\left[\alpha_k \cos\left(\frac{2k+1}{2}\pi t\right) + \beta_k \sin\left(\frac{2k+1}{2}\pi t\right)\right]$

where $\alpha_k = 2\,\mathrm{ic}_1(c_3 + c_4)$, and $\beta_k = -2c_1(c_3 - c_4)$. For c_1 nonzero and c_3, c_4 are not all zero.

#2 solution:

Step 1: Find separable solutions:

Set $u(x,t) = X(x)T(t)$, by the same argument we get solutions:

If $\sigma = 0$, then $u(x,t) = (c_1 + c_2 x)(c_3 + c_4 t)$.

If $\sigma \neq 0$, then $u(x,t) = \left(c_1 e^{\sqrt{\sigma}x} + c_2 e^{-\sqrt{\sigma}x}\right)\left(c_3 e^{3\sqrt{\sigma}t} + c_4 e^{-3\sqrt{\sigma}t}\right)$.

Step 2: use the boundary conditions to get σ:

If $\sigma = 0$, by the same argument, the only solution is trivial.

If $\sigma \neq 0$, then by $u(0,t) = 0$, we have $c_1 + c_2 = 0$, from $u(2,t) = 0$, we have $e^{4\sqrt{\sigma}} = 1$ and $\sqrt{\sigma} = \frac{k\pi}{2}i$

$\sigma = -\frac{k^2\pi^2}{4}$.

Hence the separable solutions are

$u(x,t) = c_1 \left(e^{i\frac{k\pi}{2}x} - e^{-i\frac{k\pi}{2}x}\right)\left(c_3 e^{i\frac{3k\pi}{2}t} + c_4 e^{-i\frac{3k\pi}{2}t}\right) = \sin\left(\frac{k\pi}{2}x\right)\left[\alpha_k \cos\left(\frac{3k\pi}{2}t\right) + \beta_k \sin\left(\frac{3k\pi}{2}t\right)\right]$.

Step 3: the general solution is now

$u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{2}x\right)\left[\alpha_k \cos\left(\frac{3k\pi}{2}t\right) + \beta_k \sin\left(\frac{3k\pi}{2}t\right)\right]$.

By condition $u(x,0) = \sin(\pi x) + 2\sin(5\pi x) = \sum_{k=1}^{\infty} \alpha_k \sin\left(\frac{k\pi}{2}x\right)$, we have $\alpha_2 = 1, \alpha_{10} = 2$ and others are zero.

By condition $u_t(x,0) = 1 = \sum_{k=1}^{\infty} \frac{3k\pi}{2}\beta_k \sin\left(\frac{k\pi}{2}x\right)$, we have

$\beta_k = \frac{2}{3k\pi}\frac{2}{2}\int_0^2 \sin\left(\frac{k\pi}{2}x\right) \mathrm{dx} = \frac{4}{3k^2\pi^2}\left(1 - (-1)^k\right)$.