Math 267 : Exercises 1 Solutions

All questions are due **Monday January 14th**

1. Simplify each number to the form $a + ib$

   (a) $\frac{1}{3-i}$

   (b) $\frac{4+i}{2-3i}$

   (c) $\frac{(12+5i)^2}{2-4i}$

   Answer

   (a) $\frac{1}{3-i} = \frac{1}{3-i}\frac{3+i}{3+i} = \frac{3+i}{3^2+1^2} = \frac{3}{10} + i\frac{1}{10}$

   (b) $\frac{4+i}{2-3i} = \frac{4+i}{2-3i}\frac{2+3i}{2+3i} = \frac{5}{13} + i\frac{14}{13}$

   (c) $\frac{(12+5i)^2}{2-4i} = \frac{119+120i}{2-4i} = -\frac{121}{10} + i\frac{179}{5}$

2. Find **all** complex numbers z with $z^6 = 27\,i$.

   Answer

   Substitute $z = r\,e^{i\theta} \longrightarrow r^6\,e^{i6\theta} = 27\,i = 27\,e^{i\frac{\pi}{2}}$.

   Match $r^6 = 27$, gives $r = \sqrt{3}$.

   Match $e^{i6\theta} = e^{i\frac{\pi}{2}}$, gives $6\theta = \frac{\pi}{2}$, $6\theta = \frac{\pi}{2} + 2\pi$, ..., $6\theta = \frac{\pi}{2} + 10\pi$.

   $z = \sqrt{3}e^{i\frac{\pi}{12}}$, $\sqrt{3}e^{i(\frac{\pi}{12}+\frac{\pi}{3})}$, $\sqrt{3}e^{i(\frac{\pi}{12}+\frac{2\pi}{3})}$, $\sqrt{3}e^{i(\frac{\pi}{12}+\pi)}$, $\sqrt{3}e^{i(\frac{\pi}{12}+\frac{4\pi}{3})}$, and $\sqrt{3}e^{i(\frac{\pi}{12}+\frac{5\pi}{3})}$.

3. Use Euler's theorem to rewrite $\sum_{k=-\infty}^{+\infty} \frac{\sin(kx)}{k^2+1}$ as a sum of complex exponentials.

   Answer Note that $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$. Therefore,

$$
\begin{aligned}
\sum_{k=-\infty}^{+\infty} \frac{\sin(kx)}{k^2+1} &= \sum_{k=-\infty}^{+\infty} \frac{e^{ikx} - e^{-ikx}}{2i(k^2+1)} \\
&= \sum_{k=-\infty}^{+\infty} \frac{e^{ikx}}{2i(k^2+1)} - \sum_{k=-\infty}^{+\infty} \frac{e^{-ikx}}{2i(k^2+1)} \\
&= \sum_{k=-\infty}^{+\infty} \frac{e^{ikx}}{2i(k^2+1)} - \sum_{l=-\infty}^{+\infty} \frac{e^{ilx}}{2i((-l)^2+1)} \quad \text{letting } l = -k \\
&= \sum_{k=-\infty}^{+\infty} \frac{e^{ikx}}{2i(k^2+1)} - \sum_{k=-\infty}^{+\infty} \frac{e^{ikx}}{2i(k^2+1)} \\
&= 0
\end{aligned}
$$

4. Substitute $u(x,t) = X(x)T(t)$ into,

$$\begin{cases} \partial_t u = \partial_x^2 u - \partial_x u \\ u(0,t) = u(5,t) = 0 \end{cases}$$

Find an ODE for $X(x)$, and another for $T(t)$. Do not solve the ODEs.

Note: you should have boundary conditions for $X(x)$.

Answer

$$X(x)\,T'(t) = X''(x)\,T(t) - X'(x)\,T(t) \Rightarrow \quad \frac{T'(t)}{T(t)} = \frac{X''(x) - X'(x)}{X(x)}$$

Both sides must equal a constant $-\sigma$. Now, from the constraints:

$$u(0,t) = X(0)T(t) = 0 \Rightarrow \quad X(0) = 0$$

Together:

$$\begin{cases} X'' - X' + \sigma X = 0 \\ X(0) = 0 \\ X(5) = 0, \end{cases}$$

where σ is an unknown constant. For $T(t)$, we have the ODE:

$$T'(t) + \sigma T(t) = 0.$$

with no initial condition provided.

5. Find **all** numbers σ such that,

$$\begin{cases} -\sigma = \frac{\partial_x^2 X(x)}{X(x)} \\ X(0) = 0 \\ \partial_x X(1) = 0 \end{cases}$$

has a solutions $X(x) \not\equiv 0$.

Answer

$X'' + \sigma X = 0$ has characteristic polynomial $r^2 + \sigma = 0$.

Case 1 : $\sigma < 0$, eg: $r = \pm\sqrt{-\sigma}$

$X(x) = c_1 e^{\sqrt{-\sigma}x} + c_2 e^{-\sqrt{-\sigma}x}$

$$\left.\begin{array}{l} 0 = X(0) = c_1 + c_2 \\ 0 = X'(1) = c_1\sqrt{-\sigma}e^{\sqrt{-\sigma}} - c_2\sqrt{-\sigma}e^{-\sqrt{-\sigma}} \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = -c_2, \\ c_1\sqrt{-\sigma}\left(e^{\sqrt{-\sigma}} + e^{-\sqrt{-\sigma}}\right) = 0 \end{array}$$

This means, $c_1 = c_2 = 0$, so the only solution is $X(x) = 0$.

Case 2 : $\sigma = 0$, eg: $r = 0$ twice

$X(x) = c_1 + c_2 x$

$0 = X(0) = c_1$, and $0 = X'(1) = c_2$, so the only solution is $X(x) = 0$.

Case $3:$ $\sigma > 0$, eg: $r = \pm i\sqrt{\sigma}$

$X(x) = c_1 \cos(\sqrt{\sigma}x) + c_2 \sin(\sqrt{\sigma}x)$

$$\left.\begin{array}{l} 0 = X(0) = c_1 \\ 0 = X'(1) = 0 + c_2\sqrt{\sigma}\cos(\sqrt{\sigma}) \end{array}\right\} \Rightarrow X(x) = 0 \text{ or } \cos(\sqrt{\sigma}) = 0$$

This means $\sqrt{\sigma} = \frac{\pi}{2} + k\pi$, where k is an integer.

$\sigma = \pi^2\left(\frac{1}{2}\right)^2, \pi^2\left(\frac{3}{2}\right)^2, \pi^2\left(\frac{5}{2}\right)^2, \ldots$

6. Fully solve:

$$\begin{cases} \partial_t^2 u = 16\partial_x^2 u \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 0 \\ \partial_t u(x,0) = \cos(x)\sin(x) \end{cases}$$

Use the general solution formula from lecture.

Answer

The general solution (for the given wave equation and the boundary conditions) is

$$u(x,t) = \sum_{k=1}^{+\infty} \sin(kx)\left(\alpha_k \cos(4kt) + \beta_k \sin(4kt)\right)$$

Now, use the initial conditions. From

$$0 = u(x,0) = \sum_{k=1}^{\infty} \alpha_k \sin(kx)$$

we see $\alpha_k = 0$ for $k = 1, 2, 3, \cdots$.. To use $\partial_t u(x,0) = \cos(x)\sin(x)$ first notice that $\sin(2x) = 2\sin x \cos x$. So, we get

$$\frac{1}{2}\sin(2x) = u_t(x,0) = \sum_{k=1}^{\infty} \beta_k 4k \sin(kx)$$

Therefore, for $k = 2$ $8\beta_2 = \frac{1}{2}$, so $\beta_2 = \frac{1}{16}$, and for $k \neq 2$, $b_k = 0$.

Finally, the solution is

$$u(x,t) = \frac{1}{16}\sin(2x)\sin(8t).$$

Alternative (but, equivalent) solution:

Recognize constants $c_0 = 4$ and $L = \pi$. General solution is then:

$$u(x,t) = \sum_{k=1}^{+\infty} \left(e^{ikx} - e^{-ikx}\right) \left(\alpha_k\, e^{i4kt} + \beta_k\, e^{-i4kt}\right)$$

Using first piece of data:

$$0 = u(x,0) = \sum_{k=1}^{+\infty} \left(e^{ikx} - e^{-ikx}\right) (\alpha_k + \beta_k)$$

That means $\alpha_k + \beta_k = 0$, for every k.

Using second piece of data:

$$\cos(x)\sin(x) = \sum_{k=1}^{+\infty} \left(e^{ikx} - e^{-ikx}\right) (\alpha_k\,(i4k) + \beta_k\,(-i4k))$$

$$\cos(x)\sin(x) = \sum_{k=1}^{+\infty} \left(e^{ikx} - e^{-ikx}\right) (i4k)\,(\alpha_k - \beta_k)$$

$$\frac{e^{ix} + e^{-ix}}{2}\frac{e^{ix} - e^{-ix}}{2i} = \sum_{k=1}^{+\infty} \left(e^{ikx} - e^{-ikx}\right) (i4k)\,(\alpha_k - \beta_k)$$

$$\frac{1}{4i}\left(e^{i2x} - e^{-i2x}\right) = \sum_{k=1}^{+\infty} \left(e^{ikx} - e^{-ikx}\right) (i4k)\,(\alpha_k - \beta_k)$$

Note, the term on the left shows up when $k = 2$.

That means: $\frac{1}{4i} = (i\,4\cdot 2)\,(\alpha_2 - \beta_2)$, and $0 = (i4k)\,(\alpha_k - \beta_k)$ for all $k \neq 2$.

$$\left.\begin{array}{l} \alpha_k = -\beta_k \text{ for all } k \\ \alpha_2 = \beta_2 - \dfrac{1}{32} \\ \alpha_k = +\beta_k \text{ for all } k \neq 2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \alpha_2 = -\dfrac{1}{64} \\ \beta_2 = +\dfrac{1}{64} \\ \alpha_k = \beta_k = 0 \text{ for all } k \neq 2 \end{array}\right.$$

$$u(x,t) = \left(e^{i2x} - e^{-i2x}\right)\left(-\frac{1}{64}\, e^{i4\cdot 2t} + \frac{1}{64}\, e^{-i4\cdot 2t}\right)$$