Math 267, Section 202 : HW 10 Solutions

Questions will **not** be collected. Solutions will be posted shortly.

1. Use the definition to calculate the z-transform and region of convergence of each signal. Sketch the region of convergence, and mark the poles.

   (a) $x[n] = \left(\frac{1}{5}\right)^n u[n] + 2^n u[-n-1]$

   (b) $y[n] = (3-4i)^n u[n]$

   (c) $a[n] = \left(\frac{1}{3}\right)^{|n|}$

   (d) $b[n] = 3^{n+2} u[n-8]$

   Answer

   (a)

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{5}\right)^n u[n] + 2^n u[-n-1] \right) z^{-n} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n \cdot 1 \cdot z^{-n} + \sum_{n=-\infty}^{-1} 2^n \cdot 1 \cdot z^{-n} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n \cdot 1 \cdot z^{-n} + \sum_{\ell=0}^{\infty} 2^{-\ell-1} \cdot 1 \cdot z^{-(-\ell-1)} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{5z}\right)^n + \frac{z}{2} \sum_{\ell=0}^{\infty} \left(\frac{z}{2}\right)^\ell \\
&= \frac{1}{1-\frac{1}{5z}} + \frac{z}{2} \frac{1}{1-\frac{z}{2}},
\end{aligned}$$

   where the geometric sums require $\left|\frac{1}{5z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$. That is: $\frac{1}{5} < |z|$ and $|z| < 2$.
   The region of convergence is the annulus $\frac{1}{5} < z < 2$, and there are poles are $+\frac{1}{5}$ and $+2$.

   (b)

$$\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} (3-4i)^n u[n] z^{-n} \\
&= \sum_{n=0}^{\infty} \left(\frac{3-4i}{z}\right)^n \\
&= \frac{1}{1-\frac{3-4i}{z}},
\end{aligned}$$

   where the geometric sum requires $\left|\frac{3-4i}{z}\right| < 1$. That is $5 = |3-4i| < z$.

The region of convergence is outside a circle, $5 < |z|$, and there is a pole at $+3 - 4i$ (in the fourth quadrant).

(c)

$$\begin{aligned} A(z) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^{|n|} z^{-n} \\ &= \sum_{n=-\infty}^{-1} \left(\frac{1}{3}\right)^{-n} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n} z^{-n} \\ &= \frac{z}{3} \sum_{\ell=0}^{\infty} \left(\frac{z}{3}\right)^{\ell} + \sum_{n=0}^{\infty} \left(\frac{1}{3z}\right)^{n} \\ &= \frac{z}{3} \frac{1}{1-\frac{z}{3}} + \frac{1}{1-\frac{1}{3z}}, \end{aligned}$$

where geometric sums require $\left|\frac{z}{3}\right| < 1$ and $|\frac{1}{3z}| < 1$.
The region of convergence is thus $1/3 < |z| < 3$, and there are poles at $z = +3$ and $z = +\frac{1}{3}$.

(d)

$$\begin{aligned} B(z) &= \sum_{n=-\infty}^{\infty} 3^{n+2} u[n-8] z^{-n} \\ &= \sum_{n=8}^{\infty} 3^{n+2}\, z^{-n} \\ &= \sum_{\ell=0}^{\infty} 3^{\ell+10}\, z^{-\ell-8}, \qquad \text{where we used } \ell = n-8 \\ &= \frac{3^{10}}{z^8} \sum_{\ell=0}^{\infty} \left(\frac{3}{z}\right)^{\ell} \\ &= \frac{3^{10}}{z^8} \frac{1}{1-\frac{3}{z}} \end{aligned}$$

where the geometric sum requires $\left|\frac{3}{z}\right| < 1$.
The region of convergence is outside a circle, $3 < |z|$, and there is a pole at $z = +3$.

2. In each case, invert the z-transform to find $x[n]$

   (a) $X(z) = \frac{z^2}{z^2+z+1-i}$ and ROC $|z| < 1$

   (b) $X(z) = \frac{z}{(z-3)(z^2-i\,z+\frac{3}{4})}$ and ROC $\frac{3}{2} < |z| < 3$

   Answer

   (a) Factoring and partial fractions:

$$X(z) = \frac{z}{z-i} \frac{z}{z+1+i} = \frac{2+i}{5} \frac{z}{z-i} + \frac{3-i}{5} \frac{z}{z+1+i}$$

To get the correct ROC:

$$x[n] = -\frac{2+i}{5} \cdot (i)^n u[-n-1] - \frac{3-i}{5} \cdot (-1-i)^n u[-n-1]$$

(b) Factoring and partial fractions:

$$X(z) = \frac{z}{z-3} \frac{1}{z+\frac{i}{2}} \frac{1}{z-\frac{3i}{2}} = a\frac{z}{z-3} + b\frac{z}{z+\frac{i}{2}} + c\frac{z}{z-\frac{3i}{2}},$$

where $a = \frac{52+16i}{555}$, $b = \frac{-1-6i}{37}$ and $c = \frac{-1+2i}{15}$. To get the correct ROC:

$$x[n] = -a \cdot 3^n u[-n-1] + b\left(-\frac{i}{2}\right)^n u[n] + c\left(\frac{3i}{2}\right)^n u[n]$$

eg: $\frac{3}{2} < |z| < 3$ is the overlap of $|z| < 3$, $\frac{1}{2} < |z|$ and $\frac{3}{2} < |z|$.

3. Calculate the following convolutions, *using the z-transform.*

(a) $(x * y)[n]$ when $x[n] = 2^n u[n]$ and $y[n] = \frac{1}{4}^n u[n+3]$

(b) $(x * y)[n]$ when $x[n] = u[1-n]$ and $y[n] = (-1)^n u[-n]$

Can you do the same thing using DTFT?

Answer

(a) Let $a[n] = 2^n u[n]$ and $b[n] = \frac{1}{4}^n u[n+3] = \frac{1}{4}^{-3} \frac{1}{4}^{n+3} u[n+3] = 4^3 \frac{1}{4}^{n+3} u[n+3]$. Then $A(z) = \frac{z}{z-2}$ with $|z| > 2$, and, using the time-shift property, $B(z) = 4^3 z^3 \frac{z}{z-\frac{1}{4}}$ with $|z| > \frac{1}{4}$. Using the convolution property, the z-trans of $a[n] * b[n]$ is:

$$4^3 \frac{z}{z-2} z^3 \frac{z}{z-\frac{1}{4}}, \qquad \text{with } |z| > 2 \text{ (the intersection)}$$

By partial fractions:

$$4^3 \frac{z}{z-2} z^3 \frac{z}{z-\frac{1}{4}} = 4^3 \left[\frac{8}{7} z^3 \frac{z}{z-2} - \frac{1}{7} z^3 \frac{z}{z-\frac{1}{4}}\right]$$

Using the time-shift property, and to get the ROC of $|z| > 2$, this is the z-transform of:

$$\underline{4^3 \left[\frac{8}{7} 2^{(n+3)} u[(n+3)] - \frac{1}{7}\left(\frac{1}{4}\right)^{(n+3)} u[(n+3)]\right]}$$

Since this signal and the convolution have the same z-transform, they must be the same signal.

(b) Let $a[n] = u[1-n] = u[-(n-2)-1]$ and $b[n] = (-1)^n u[-n] = -(-1)^{(n-1)} u[-(n-1)-1]$. Then $A(z) = -z^{-2}\frac{z}{z-1}$ with $|z| < 1$, and $B(z) = +z^{-1}\frac{z}{z+1}$ also with $|z| < 1$. The convolution has z-transform:

$$-z^{-2}\frac{z}{z-1}z^{-1}\frac{z}{z+1} = -\frac{1}{2}z^{-3}\frac{z}{z-1} - \frac{1}{2}z^{-3}\frac{z}{z+1}$$

Another signal with the same z-transform formula, and same ROC of $|z| < 1$ is:

$$+\frac{1}{2}\,1^{n-3}u[-(n-3)-1] + \frac{1}{2}\,(-1)^{n-3}u[-(n-3)-1]$$

There is a problem using DTFT here, since for signals $2^n u[n], u[1-n], (-1)^n u[-n]$, corresponding DTFTs do not exist since the corresponding infinite summations do not converge.

4. Suppose an LTI system

$$y[n] - 3y[n-1] = x[n]$$

is *causal*.

(a) Find $y[n]$ for $x[n] = 5^n u[n]$.

(b) Find $y[n]$ for $x[n] = -5^n u[-n]$.

Answer:

(a) Take the z-transform of both sides,

$$Y(z) - 3z^{-1}Y(z) = X(z),$$

and rearrange:

$$Y(z) = \frac{z}{z-3}X(z) = H(z)\,X(z)$$

Note that since the system is causal, for $H(z) = \frac{z}{z-3}$, ROC is $|z| > 3$. Here, $X(z) = \frac{z}{z-5}$ with ROC $|z| > 5$. For $H(z)X(z) = \frac{z}{z-3}\frac{z}{z-5}$, the ROC is given by the intersection, so it is $|z| > 5$. By partial fractions,

$$Y(z) = \frac{z}{z-3}\frac{z}{z-5} = z^2\frac{1}{2}\Big(\frac{1}{z-5} - \frac{1}{z-3}\Big)$$
$$= z\frac{1}{2}\Big(\frac{z}{z-5} - \frac{z}{z-3}\Big)$$

To get the desired ROC (and using the time-shift property), we must have:

$$y[n] = \underline{\frac{1}{2}\Big(5^{n+1}u[n+1] - 3^{n+1}u[n+1]\Big)}.$$

(b) Take the z-transform of both sides,

$$Y(z) - 3z^{-1}Y(z) = X(z),$$

and rearrange:

$$Y(z) = \frac{z}{z-3}X(z) = H(z)\,X(z)$$

Note that since the system is causal, for $H(z) = \frac{z}{z-3}$, ROC is $|z| > 3$. Here, since $x[n] = -5^n u[-n] = -5 \cdot 5^{n-1}u[-(n-1)-1]$, by time-shifting property $X(z) = 5z^{-1}\frac{z}{z-5} = 5\frac{1}{z-5}$ with ROC $|z| < 5$. For $H(z)X(z) = 5\frac{z}{z-3}\,\frac{1}{z-5}$, the ROC is given by the intersection, so it is $3 < |z| < 5$. By partial fractions,

$$\begin{aligned} Y(z) &= 5\frac{z}{z-3}\,\frac{1}{z-5} = z\frac{5}{2}\Big(\frac{1}{z-5} - \frac{1}{z-3}\Big) \\ &= \frac{5}{2}\Big(\frac{z}{z-5} - \frac{z}{z-3}\Big) \end{aligned}$$

To get the desired ROC, we must have:

$$y[n] = \underline{\frac{5}{2}\Big(- 5^n u[-n-1] - 3^n u[n]\Big)}.$$

5. Consider the LTI given by the difference equation:

$$6\,y[n+2] - y[n+1] - y[n] = x[n]$$

(a) Find the system function $H(z)$. (The system function is by definition, the z-transform of the impulse response function $h[n]$.)

(b) Plot the poles of $H(z)$, and state the *possible* regions of convergence.

(c) Find the impulse response $h[n]$ that is *neither left or right-sided.* (Here, $h[n]$ is called left-sided if $h[n] = 0$ for all large positive number n, i.e. $n \geq n_0$ for a positive number n_0, and it is called right-side if $h[n] = 0$ for all sufficiently negative numbers n, i.e. $n < n_0$ for a negative number n_0.

(d) When the impulse response function $h[n]$ is given from part(c), calculate the output for the input $x[n] = u[n+4] - u[n]$.

Answer

(a) Take the z-transform of both sides, and rearrange:

$$Y(z) = \frac{1}{6z^2 - z - 1}X(z) = H(z)\,X(z)$$

(b) Factoring, $H(z) = \frac{1}{6}\frac{1}{z+\frac{1}{3}}\,\frac{1}{z-\frac{1}{2}}$. There is a pole at $-\frac{1}{3}$ and another at $+\frac{1}{2}$. Possible ROC are: $|z| < \frac{1}{3}$, $\frac{1}{3} < |z| < \frac{1}{2}$, and $\frac{1}{2} < |z|$.

(c) Only the ROC $\frac{1}{3} < |z| < \frac{1}{2}$ corresponds to **neither** left or right-sided. By partial fractions,

$$H(z) = \frac{1}{6} \frac{1}{z+\frac{1}{3}} \frac{1}{z-\frac{1}{2}} = -\frac{1}{5} \frac{1}{z} \frac{z}{z+\frac{1}{3}} + \frac{1}{5} \frac{1}{z} \frac{z}{z-\frac{1}{2}}$$

To get the desired ROC (and using the time-shift property), we must have:

$$\begin{aligned} h[n] &= -\frac{1}{5}\left(-\frac{1}{3}\right)^{(n-1)} u[(n-1)] + \frac{1}{5}(-1)\left(\frac{1}{2}\right)^{(n-1)} u[-(n-1)-1] \\ &= \frac{3}{5}\left(\frac{-1}{3}\right)^{n} u[n-1] - \frac{2}{5}\left(\frac{1}{2}\right)^{n} u[-n] \end{aligned}$$

(d) Note that $y[n] = (h * x)[n]$. One may try to use z-tranform, but, in this case, note that the ROC is the intersection of ROC of $X(z)$ and that of $H(z)$, so the intersection of $|z| > 1$ and $\frac{1}{3} < |z| < \frac{1}{2}$, therefore, the ROC for $Y(z)$ is an empty set. This means the z-transform method does not apply in this case.

Therefore, we apply the definition of the convolution:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} \left(\frac{3}{5}\left(\frac{-1}{3}\right)^{k} u[k-1] - \frac{2}{5}\left(\frac{1}{2}\right)^{k} u[-k]\right)(u[(n-k)+4] - u[n-k]) \\ &= \sum_{k=n+1}^{n+4} \left(\frac{3}{5}\left(\frac{-1}{3}\right)^{k} u[k-1] - \frac{2}{5}\left(\frac{1}{2}\right)^{k} u[-k]\right) \\ &= \sum_{k=n+1}^{n+4} \frac{3}{5}\left(\frac{-1}{3}\right)^{k} u[k-1] - \sum_{k=n+1}^{n+4} \frac{2}{5}\left(\frac{1}{2}\right)^{k} u[-k] \end{aligned}$$

There are **many** cases here; each case takes a couple lines, and a geometric sum or two.

Let us first handle $\sum_{k=n+1}^{n+4} \frac{3}{5}\left(\frac{-1}{3}\right)^{k} u[k-1]$.

Case $n+4 < 1$, i.e. $n \le -4$: the sum is zero.

Case $n+1 < 1 \le n+4$, i.e. $n = -1, -2, -3$: the sum reduces to $\sum_{k=1}^{n+4} \frac{3}{5}\left(\frac{-1}{3}\right)^{k} = \frac{3}{5}\left(\frac{-1}{3}\right)\frac{1-\left(-\frac{1}{3}\right)^{n+4}}{1-\left(-\frac{1}{3}\right)} = -\frac{3}{20}(1-(-\frac{1}{3})^{n+1})$.

Case $1 \le n+1$, i.e. $n \ge 0$: the sum reduces to $\sum_{k=n+1}^{n+4} \frac{3}{5}\left(\frac{-1}{3}\right)^{k} = \frac{3}{5}\left(\frac{-1}{3}\right)^{n+1}\frac{1-\left(-\frac{1}{3}\right)^{4}}{1-\left(-\frac{1}{3}\right)} = \frac{4}{9}(-\frac{1}{3})^{n+1}$.

Now, let us handle $\sum_{k=n+1}^{n+4} \frac{2}{5}\left(\frac{1}{2}\right)^{k} u[-k]$.

Case $n+1 > 0$, i.e. $n > -1$: the sum is zero.

Case $n+1 \le 0 < n+4$, i.e. $n = -1, -2, -3$: the sum reduces to $\sum_{k=n+1}^{0} \frac{2}{5}\left(\frac{1}{2}\right)^{k} = \frac{2}{5}\left(\frac{1}{2}\right)^{n+1}\frac{1-\left(-\frac{1}{2}\right)^{-(n+1)}}{1-\frac{1}{2}} = \frac{4}{5}((\frac{1}{2})^{n+1} - (-1)^{n+1})$.

Case $n+4 \leq 0$, i.e. $n \leq -4$: the sum reduces to $\sum_{k=n+1}^{n+4} \frac{2}{5}\left(\frac{1}{2}\right)^k = \frac{3}{5}\left(\frac{1}{2}\right)^{n+1} \frac{1-\left(\frac{1}{2}\right)^4}{1-\frac{1}{2}} = \frac{9}{16} \cdot \left(\frac{1}{2}\right)^n$.

Combining all these, we get

$$y[n] = \begin{cases} \frac{4}{9} \cdot \left(-\frac{1}{3}\right)^{n+1} & \text{for } n \geq 0 \\ -\frac{3}{20}\left(1 - (-\frac{1}{3})^{n+1}\right) - \frac{4}{5}\left((\frac{1}{2})^{n+1} - (-1)^{n+1}\right). & \text{for } n = -1, -2, -3 \\ -\frac{9}{16} \cdot \left(\frac{1}{2}\right)^n & \text{for } n \leq -4 \end{cases}$$

6. Consider the LTI given by the difference equation:

$$y[n+1] - 2y[n] + y[n-1] = i\, y[n-1] + x[n+1]$$

(a) Find the system function $H(z)$. (The system function is by definition, the z-transform of the impulse response function $h[n]$.)

(b) Plot the poles of $H(z)$, and state the *possible* regions of convergence.

(c) Find the impulse response $h[n]$ that makes the LTI *causal.*

(d) Calculate the output when $x[n] = \delta_{-1}[n] + \delta_2[n]$

Answer

(a)

$$\left(z - 2 + z^{-1}\right) Y(z) = iz^{-1}\, Y(z) + z\, X(z) \Rightarrow \quad Y(z) = \frac{z}{z - 2 + (1-i)z^{-1}} X(z)$$

(b) Factoring and using partial fractions:

$$\begin{aligned} H(z) = \frac{z^2}{z^2 - 2z + (1-i)} &= \frac{z^2}{(z - 1 + \frac{1+i}{2})(z - 1 - \frac{1+i}{2})} \\ &= \frac{-1+i}{2} \frac{z^2}{z - 1 + \frac{1+i}{2}} + \frac{1-i}{2} \frac{z^2}{z - 1 - \frac{1+i}{2}} \end{aligned}$$

There are poles at $+\frac{1}{2} + \frac{i}{2}$ and $+\frac{3}{2} - \frac{i}{2}$, in the first and fourth quadrants, respectively. The possible regions of convergence are: $|z| < \frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}} < |z| < \frac{\sqrt{10}}{2}$ and $\frac{\sqrt{10}}{2} < |z|$. Only the last ROC is *causal.*

(c) The strictly right-sided signal giving the formula for $H(z)$ is:

$$h[n] = \Big[\frac{-1+i}{2}\left(\frac{1-i}{2}\right)^{n+1} + \frac{1-i}{2}\left(\frac{3+i}{2}\right)^{n+1}\Big] u[n+1]$$

(d) As always, $y[n] = h[n] * x[n]$, and convolving by delta shifts time:

$$y[n] = h[n+1] + h[n-2]$$

that is,

$$\frac{-1+i}{2}\left(\frac{1-i}{2}\right)^{n+2} u[n+2] + \frac{1-i}{2}\left(\frac{3+i}{2}\right)^{n+2} u[n+2]$$
$$+ \frac{-1+i}{2}\left(\frac{1-i}{2}\right)^{n-1} u[n-1] + \frac{1-i}{2}\left(\frac{3+i}{2}\right)^{n-1} u[n-1]$$