Identities

$z = x + iy \qquad z = |z|\, e^{i\theta} \qquad |z| = \sqrt{z\overline{z}} = \sqrt{x^2+y^2}$

$\overline{z} = x - iy \qquad \arg(z) = \theta = \arctan\left(\frac{y}{x}\right)$

$$\sum_{k=0}^{N-1} r^k = \frac{1-r^N}{1-r}, \quad \text{when } r \neq +1$$

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

$e^{i\,k\,\pi} = (-1)^k \qquad e^{i\,k\,\frac{\pi}{2}} = i^k$

Fourier Series – for f(x) with period $\mathtt{T} = 2L$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi}{L}x\right) + b_k \sin\left(\frac{k\pi}{L}x\right)$$

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi}{L}x\right), \quad b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi}{L}x\right)$$

$$f(x) = c_0 + \sum_{k\neq 0}^{\infty} c_k\, e^{i\,k\,\frac{2\pi}{\mathtt{T}}x}$$

$$c_k = \frac{1}{\mathtt{T}}\int_a^{a+\mathtt{T}} f(x)\overline{e^{i\,k\,\frac{2\pi}{\mathtt{T}}x}}dx$$

Sine Series – for f(x), 0<x<L

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}x\right) \qquad b_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi}{L}x\right)dx$$

$$f(x) = \sum_{k=1}^{\infty} c_k\left(e^{i\frac{k\pi}{L}x} - e^{-i\frac{k\pi}{L}x}\right) \qquad c_k = \frac{1}{2L}\int_0^L f(x)\left(\overline{e^{i\frac{k\pi}{L}x} - e^{-i\frac{k\pi}{L}x}}\right)dx$$

Cosine Series – for f(x), 0<x<L

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi}{L}x\right) \qquad a_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{k\pi}{L}x\right)dx$$

$$f(x) = c_0 + \sum_{k=1}^{\infty} c_k\left(e^{i\frac{k\pi}{L}x} + e^{-i\frac{k\pi}{L}x}\right) \qquad c_k = \frac{1}{2L}\int_0^L f(x)\left(\overline{e^{i\frac{k\pi}{L}x} + e^{-i\frac{k\pi}{L}x}}\right)dx$$

Wave Equation $u_{tt} = c^2 u_{xx}$, when u(0,t) = u(L,t) = 0

$$u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{L}x\right)\left(\alpha_k \sin\left(c\frac{k\pi}{L}t\right) + \beta_k\cos\left(c\frac{k\pi}{L}t\right)\right)$$

$$u(x,t) = \sum_{k=1}^{\infty}\left(e^{i\frac{k\pi}{L}x} - e^{-i\frac{k\pi}{L}x}\right)\left(\alpha_k e^{ic\frac{k\pi}{L}t} + \beta_k e^{-ic\frac{k\pi}{L}t}\right)$$

Heat Equation $u_t = c^2 u_{xx}$, when u(0,t) = u(L,t) = 0

$$u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{L}x\right)\ \alpha_k\, e^{-\left(c\frac{k\pi}{L}\right)^2 t}$$

$$u(x,t) = \sum_{k=1}^{\infty}\left(e^{i\frac{k\pi}{L}x} - e^{-i\frac{k\pi}{L}x}\right)\ \alpha_k\, e^{-\left(c\frac{k\pi}{L}\right)^2 t}$$

Math 267 Formula Sheet for Midterm 2

Property	*Signal*	*Fourier Transform*
	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{x}(\omega)e^{i\omega t}\,d\omega$	$\hat{x}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,dt$
	$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{y}(\omega)e^{i\omega t}\,d\omega$	$\hat{y}(\omega) = \int_{-\infty}^{\infty} y(t)e^{-i\omega t}\,dt$
Linearity	$Ax(t) + By(t)$	$A\hat{x}(\omega) + B\hat{y}(\omega)$
Time shifting	$x(t - t_0)$	$e^{-i\omega t_0}\hat{x}(\omega)$
Frequency shifting	$e^{i\omega_0 t}x(t)$	$\hat{x}(\omega - \omega_0)$
Scaling	$x\left(\frac{t}{\alpha}\right)$	$\lvert\alpha\rvert\hat{x}(\alpha\omega)$
Time shift & scaling	$x\left(\frac{t-t_0}{\alpha}\right)$	$\lvert\alpha\rvert e^{-i\omega t_0}\hat{x}(\alpha\omega)$
Frequency shift & scaling	$\lvert\alpha\rvert e^{i\omega_0 t}x(\alpha t)$	$\hat{x}\left(\frac{\omega-\omega_0}{\alpha}\right)$
Conjugation	$\overline{x(t)}$	$\overline{\hat{x}(-\omega)}$
Time reversal	$x(-t)$	$\hat{x}(-\omega)$
t–Differentiation	$x'(t)$	$i\omega\hat{x}(\omega)$
	$x^{(n)}(t)$	$(i\omega)^n\hat{x}(\omega)$
ω–Differentiation	$tx(t)$	$i\frac{d}{d\omega}\hat{x}(\omega)$
	$t^n x(t)$	$\left(i\frac{d}{d\omega}\right)^n\hat{x}(\omega)$
Convolution	$\int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,d\tau$	$\hat{x}(\omega)\hat{y}(\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{x}(\theta)\hat{y}(\omega-\theta)\,d\theta$
Duality	$\hat{x}(t)$	$2\pi x(-\omega)$
Parseval	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert\hat{x}(\omega)\rvert^2\,d\omega$	
	$e^{-at}u(t) = \begin{cases} 0 & \text{if } t < 0, \\ e^{-at} & \text{if } t > 0 \end{cases}$	$\frac{1}{a + i\omega}$ (a constant, $\operatorname{Re} a > 0$)
	$e^{-at}u(-t) = \begin{cases} e^{-at} & \text{if } t < 0, \\ 0 & \text{if } t > 0 \end{cases}$	$-\frac{1}{a + i\omega}$ (a constant, $\operatorname{Re} a < 0$)
	$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2 + \omega^2}$ (a constant, $\operatorname{Re} a > 0$)
Boxcar in time	$\operatorname{rect}(t) = \begin{cases} 1 & \text{if } \lvert t\rvert < \frac{1}{2} \\ 0 & \text{if } \lvert t\rvert > \frac{1}{2} \end{cases}$	$\operatorname{sinc}\left(\frac{\omega}{2}\right) = \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right)$
General boxcar	$r_{HWC}(t) = \begin{cases} H & \text{if } \lvert t - C\rvert < \frac{W}{2} \\ 0 & \text{if } \lvert t - C\rvert > \frac{W}{2} \end{cases}$	$HWe^{-i\omega C}\operatorname{sinc}\left(\frac{W\omega}{2}\right) = e^{-i\omega C}\frac{2H}{\omega}\sin\frac{W\omega}{2}$
Boxcar in frequency	$\frac{1}{2\pi}\operatorname{sinc}\left(\frac{t}{2}\right) = \frac{1}{\pi t}\sin\left(\frac{t}{2}\right)$	$\operatorname{rect}(\omega) = \begin{cases} 1 & \text{if } \lvert\omega\rvert < 1/2 \\ 0 & \text{if } \lvert\omega\rvert > 1/2 \end{cases}$
Impulse in time	$\delta(t - t_0)$	$e^{-i\omega t_0}$
	$\delta(t - t_0)\,x(t)$	$e^{-i\omega t_0}\,x(t_0)$
Single frequency	$e^{i\omega_0 t}$	$2\pi\,\delta(\omega - \omega_0)$
Heaviside	$u(t)$	$\frac{1}{i\omega} + \pi\,\delta(\omega)$

Discrete Fourier Series (Discrete–time, period N)

Property	*Periodic Signal*	*Fourier Coefficients*
	$x[n] = \sum_{k=0}^{N-1} \hat{x}[k] e^{2\pi i \frac{kn}{N}}$ $y[n] = \sum_{k=0}^{N-1} \hat{y}[k] e^{2\pi i \frac{kn}{N}}$	$\hat{x}[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-2\pi i \frac{kn}{N}}$ $\hat{y}[k] = \frac{1}{N} \sum_{n=0}^{N-1} y[n] e^{-2\pi i \frac{kn}{N}}$
Linearity	$Ax[n] + By[n]$	$A\hat{x}[k] + B\hat{y}[k]$
Time Shifting	$x[n - n_0]$	$e^{-2\pi i \frac{kn_0}{N}} \hat{x}[k]$
Frequency Shifting	$e^{2\pi i \frac{nk_0}{N}} x[n]$	$\hat{x}[k - k_0]$
Conjugation	$\overline{x[n]}$	$\overline{\hat{x}[-k]}$
Time Reversal	$x[-n]$	$\hat{x}[-k]$
Difference	$x[n] - x[n-1]$	$\left(1 - e^{-2\pi i \frac{k}{N}}\right)\hat{x}[k]$
Convolution	$\sum_{m=0}^{N-1} x[m] y[n-m]$	$N\, \hat{x}[k]\hat{y}[k]$
Multiplication	$x[n]y[n]$	$\sum_{m=0}^{N-1} \hat{x}[m]\hat{y}[k-m]$
Duality	$\hat{x}[n]$	$\frac{1}{N} x[-k]$
Parseval	$\frac{1}{N} \sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \sum_{k=0}^{N-1} \lvert \hat{x}[k]\rvert^2$	

Discrete Time Fourier Transform (Discrete–time, aperiodic)

Property	*Aperiodic Signal*	*Fourier Transform*
	$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}(\omega) e^{i\omega n}\, d\omega$ $y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{y}(\omega) e^{i\omega n}\, d\omega$	$\hat{x}(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-i\omega n}$ $\hat{y}(\omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-i\omega n}$
Linearity	$Ax[n] + By[n]$	$A\hat{x}(\omega) + B\hat{y}(\omega)$
Time Shifting	$x[n - n_0]$	$e^{-i\omega n_0} \hat{x}(\omega)$
Frequency Shifting	$e^{i\omega_0 n} x[n]$	$\hat{x}(\omega - \omega_0)$
Conjugation	$\overline{x[n]}$	$\overline{\hat{x}(-\omega)}$
Time Reversal	$x[-n]$	$\hat{x}(-\omega)$
n–Difference	$x[n] - x[n-1]$	$\left(1 - e^{-i\omega}\right)\hat{x}(\omega)$
ω–Differentiation	$nx[n]$	$i \frac{d}{d\omega} \hat{x}(\omega)$
Convolution	$\sum_{m=-\infty}^{\infty} x[m] y[n-m]$	$\hat{x}(\omega)\hat{y}(\omega)$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}(\theta)\hat{y}(\omega - \theta)\, d\theta$
Parseval	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \lvert \hat{x}(\omega)\rvert^2\, d\omega$	

Property	*Aperiodic Signal*	*z–Transform*	*Region of Convergence*
	$x[n]$	$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$	R_X
	$y[n]$	$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n}$	R_Y
Linearity	$Ax[n] + By[n]$	$AX(z) + BY(z)$	at least $R_X \cap R_Y$
Time Shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	at least $R_X \cap \{\|z\| > 0\}$
z–scaling	$a^n x[n]$	$X\left(\frac{z}{a}\right)$	$\{\ az \mid z \text{ in } R_X\ \}$
Time Reversal	$x[-n]$	$X\left(\frac{1}{z}\right)$	$\{\ \frac{1}{z} \mid z \text{ in } R_X\ \}$
Conjugation	$\overline{x[n]}$	$\overline{X(\overline{z})}$	R_X
z–Differentiation	$nx[n]$	$-z\frac{dX}{dz}(z)$	at least the interior of R_X
Convolution	$\sum_{m=-\infty}^{\infty} x[m]y[n-m]$	$X(z)Y(z)$	at least $R_X \cap R_Y$
	$\delta[n-m] = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{z^m}$	$\|z\| > 0$
	$a^n u[n] = \begin{cases} a^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$	$\frac{1}{1-a/z} = \frac{z}{z-a}$	$\|z\| > \|a\|$
	$-a^n u[-n-1] = \begin{cases} 0 & \text{if } n \geq 0 \\ -a^n & \text{if } n < 0 \end{cases}$	$-\frac{z/a}{1-z/a} = \frac{z}{z-a}$	$\|z\| < \|a\|$
	$a^{n-1}u[n-1] = \begin{cases} a^{n-1} & n \geq 1 \\ 0 & n < 1 \end{cases}$	$\frac{1}{z-a}$	$\|z\| > \|a\|$