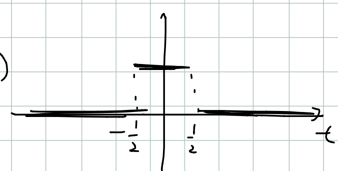
Lec 9

Fourier transform.

- motivation, formula, reason
- computation

Motivation $f(t)$ on $-\infty < t < \infty$, may not be periodic.

e.g. $\mathrm{rect}(t)$



"rectangular pulse"

$$\mathrm{rect}(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Fourier transform

$$f(t) \xrightarrow{\mathcal{F}} \hat{f}(\omega)$$

$$\boxed{\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \qquad \omega \in \mathbb{R}. \text{ "frequency variable"}}$$

"Fourier transform is like a general version of Fourier coeff. in frequency ω."

compare with

Fourier coeff. (2L-periodic case)

$$f(t) \longrightarrow C_k = \frac{1}{2L}\int_{-L}^{L} f(t)\, e^{-i\frac{k\pi}{L}t}$$

"Fourier coeff. is a function in frequency $\omega_k = \frac{k\pi}{L}$"

Ex 1

$$\widehat{\mathrm{rect}}(\omega) = \int_{-\infty}^{\infty} \mathrm{rect}(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-i\omega t}\, dt$$

Case $\omega = 0$

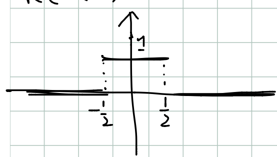
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} dt = 1$$

Case $\omega \neq 0$

$$= \frac{1}{-i\omega} e^{-i\omega t}\Big]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{-i\omega}\cdot\left(e^{-i\frac{\omega}{2}} - e^{i\frac{\omega}{2}}\right) = \frac{2}{\omega}\cdot\sin\left(\frac{\omega}{2}\right)$$

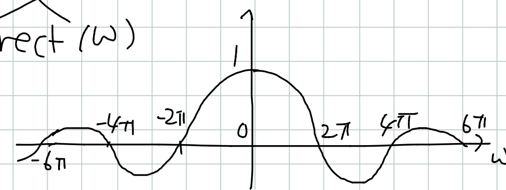
$$\therefore \quad \widehat{\text{rect}}(\omega) = \begin{cases} \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right) & \text{if } \omega \neq 0 \\ 1 & \text{if } \omega = 0 \end{cases}$$

rect(t)



$\xrightarrow{\mathcal{F}}$

$\widehat{\text{rect}}(\omega)$

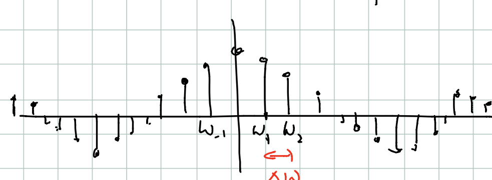


notation $\quad \text{sinc}(z) = \begin{cases} \frac{\sin(z)}{z} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases}$ $\quad \boxed{//}$

$$\boxed{\widehat{\text{rect}}(\omega) = \text{sinc}\left(\frac{\omega}{2}\right)}$$

Ex 2 $\quad f(t) = \text{rect}(t)$ on $[-L, L]$, $L \gg 1$

$$2L c_k = \int_{-L}^{L} f(t)\, e^{-i\omega_k t}\, dt = \begin{cases} \frac{2}{\omega_k}\sin\left(\frac{\omega_k}{2}\right) & \text{if } \omega_k \neq 0 \\ 1 & \text{if } \omega_k = 0 \end{cases}$$



As $L \to \infty$, $\Delta\omega = \frac{\pi}{L} \to 0$.

thus the frequencies will eventually fill the whole real line

the graph will eventually be that of $\hat{f}(\omega)$ $\boxed{//}$

Fourier inversion ("Inverse Fourier Transform")

$$\hat{f}(\omega) \xrightarrow{\mathcal{F}^{-1}} f(t)$$

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega}$$

← this integral is like Fourier series with continuous frequencies

Compare with
Fourier series (2L-periodic case) $f(t) = \sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi}{L}t}$

Reason for F.T. formula

$f(t)$ on $[-L, L]$

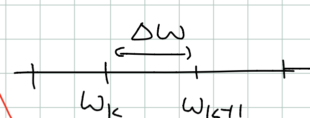
$$f(t) = \sum_{k=-\infty}^{\infty}\left[\frac{1}{2L}\int_{-L}^{L} f(s)\, e^{-i\frac{k\pi}{L}s}\, ds\right] e^{i\frac{k\pi}{L}t}$$

Let $\omega_k = \frac{k\pi}{L}$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left[\int_{-L}^{L} f(s)\, e^{-i\omega_k s}\, ds\right] e^{i\omega_k t}\, \underbrace{\frac{2\pi}{2L}}$$

$= \frac{\pi}{L} = \omega_{k+1} - \omega_k$

$= \Delta\omega$



$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left[\int_{-L}^{L} f(s)\, e^{-i\omega_k s}\, ds\right] e^{i\omega_k t}\, \Delta\omega$$

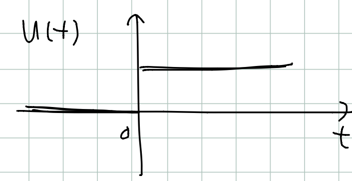
$L \to \infty \downarrow$

Therefore,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} f(s)\, e^{-i\omega s}\, ds\right]}_{\hat{f}(\omega)} e^{i\omega t}\, d\omega$$

Some Computations

unit step function $\boxed{u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}}$

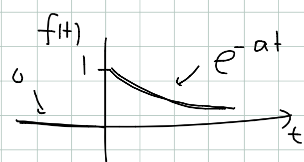


EX $a > 0$. $f(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}$

$$= e^{-at} u(t)$$

- sketch the graph
- $\hat{f}(\omega) = ?$

(sol)



$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \int_0^{\infty} e^{-at} e^{-i\omega t}\, dt$$

$$= \int_0^{\infty} e^{-(a+i\omega)t}\, dt$$

$$= \left. \frac{e^{-(a+i\omega)t}}{-(a+i\omega)} \right]_0^{\infty} = \frac{1}{a+i\omega}$$

$\boxed{/\!/}$

Note $\lim_{t \to +\infty} e^{-(a+i\omega)t} = 0$

because $\left|e^{-(a+i\omega)t}\right| = \left|e^{-at} \cdot e^{-i\omega t}\right|$

$= \left|e^{-at}\right| \left|e^{-i\omega t}\right|$ note $\theta \in \mathbb{R}$ $\left|e^{i\theta}\right| = 1$

$= \left|e^{-at}\right|$ $e^{-at} > 0$

$= e^{-at}$ $a > 0$

$\therefore \left|e^{-(a+i\omega)t}\right| \longrightarrow 0$ as $t \to +\infty$