Lec 8

- Real Fourier series vs Complex Fourier series
- Even/Odd periodic extensions
  - Fourier sine series
  - Fourier Cosine series

• Real Fourier series

2L-periodic $f(t)$ can be represented as

$$f(t) = \underline{\frac{a_0}{2} + \sum_{k=1}^{\infty} \left\{ a_k \cos\left(k\frac{\pi}{L}t\right) + b_k \sin\left(\frac{k\pi}{L}t\right) \right\}}$$

"real Fourier series"
(trigonometric)

Ex. Rewrite $\sum_{k=-\infty}^{\infty} 2^{-|k|} e^{ikt}$ as a real Fourier series

<sol> Note $e^{ikt} = \cos kt + i \sin kt$

$$\therefore \sum_{k=-\infty}^{\infty} 2^{-|k|} e^{ikt} = \sum_{k=-\infty}^{\infty} 2^{-|k|} \cos(kt) + i\, 2^{-|k|} \sin(kt)$$

$$= 2^{-0}\cos(0) + \sum_{k>0} \left[2^{-k}\cos(kt) + i\,2^{-k}\sin(kt) \right]$$

$$+ \underbrace{\sum_{k<0} \left[2^{+k}\cos(kt) + i\,2^{+k}\sin(kt) \right]}_{\text{the same as } \sum_{k>0} \left[2^{-k}\cos(-kt) + i\,2^{-k}\sin(-kt) \right]}$$

changed sign of k

$$= \sum_{k>0} \left[2^{-k}\cos(kt) - i\,2^{-k}\sin(kt) \right]$$

$\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$

$$= 1 + \sum_{k>0} \left[2^{-k} \left(\cos(kt) + \cos(kt) \right) + i 2^{-k} \left(\cancel{\sin(kt)} - \cancel{\sin(kt)} \right) \right]$$

$$= 1 + \sum_{k=1}^{\infty} 2^{-k+1} \cos(kt) \quad \square$$

In general,

$$\sum_{k=-\infty}^{\infty} c_k e^{ik\frac{\pi}{L}t} = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left\{ a_k \cos\left(k\frac{\pi}{L}t\right) + b_k \sin\left(\frac{k\pi}{L}t\right) \right\}$$

Where $a_k = c_k + c_{-k} = \frac{1}{2L} \int_{-L}^{L} f(t) \left[e^{-i\frac{k\pi}{L}t} + e^{i\frac{k\pi}{L}t} \right] dt$

$$\boxed{a_k = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(k\frac{\pi}{L}t\right) dt}$$

a_k, b_k's are real if $f(t)$ was real.

$b_k = i(c_k - c_{-k}) = \frac{1}{2L} \int_{-L}^{L} f(t)\, i \left[e^{-i\frac{k\pi}{L}t} - e^{i\frac{k\pi}{L}t} \right] dt$

$$\boxed{b_k = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{k\pi}{L}t\right) dt}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

complex F.S. $\longleftrightarrow$ real F.S.

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

If $f(t)$ even i.e. $f(t) = f(-t)$ $\Rightarrow$ $\sum_k c_k e^{ik\frac{\pi}{L}t} = \sum_k c_k e^{-ik\frac{\pi}{L}t}$

$$= \sum_k c_{-k} e^{ik\frac{\pi}{L}t}$$

(changed sign of k)

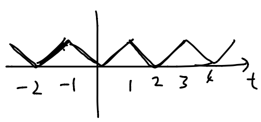
$$\Longrightarrow c_k = c_{-k}$$

$\Longrightarrow \quad b_k = 0$

$\Longrightarrow$ cosine series $\quad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi}{L}t\right)$

$$a_k = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)}_{\text{even}} \; \underbrace{\cos\left(\frac{k\pi}{L}t\right)}_{\text{even}} dt = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{k\pi}{L}t\right) dt$$

e.g. 2-periodic function $f(t)$ with $f(t) = |t|$. $-1 < t < 1$



$$f(t) = \frac{1}{2} + \sum_{k \neq 0} \frac{(-1)^k - 1}{k^2\pi^2} e^{ik\pi t} \qquad c_k = c_{-k}.$$

$$= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k - 1}{k^2\pi^2} \cdot 2 \cdot \cos(k\pi t)$$

☒

If $f(t)$ Odd i.e. $f(t) = -f(-t) \Rightarrow \sum_k c_k e^{ik\frac{\pi}{L}t} = \sum_k -c_k e^{-ik\frac{\pi}{L}t}$.

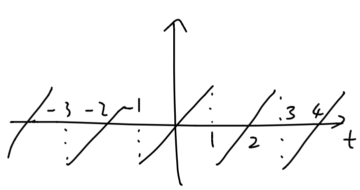
$\Longrightarrow \quad c_k = -c_{-k}$

$\Longrightarrow \quad a_k = 0$

$\Longrightarrow$ sine series $\quad \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}t\right)$

$$b_k = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(t)}_{\text{odd}} \; \underbrace{\sin\left(\frac{k\pi}{L}t\right)}_{\text{odd}}}_{\text{even}} dt = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{k\pi}{L}t\right) dt$$

e.g. 2-periodic function $f(t)$ with $f(t) = t$. $-1 < t < 1$



$$f(t) = \sum_{k \neq 0} \bar{i}\, \frac{(-1)^k}{k\;\pi^2} e^{ik\pi t} \qquad c_k = -c_{-k}$$

$$= \sum_{k=1}^{\infty} 2\, \frac{(-1)^{k+1}}{k\;\pi^2} \sin(k\pi t)$$

☒

· Periodic extension

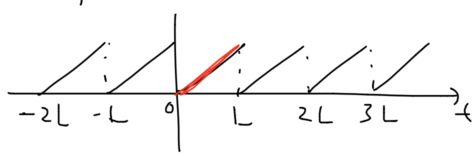
For $f(t)$ on $0<t<L$, Fourier series?

↘ whose definition required periodicity

Many ways to extend $f(t)$ to a periodic function

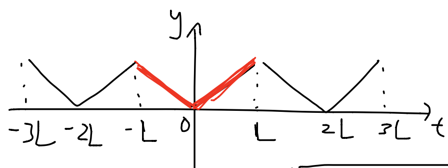
e.g. $f(t) = t$, $0<t<L$

Some periodic extensions:

∘ 

period = L

· Even extension



period = 2L

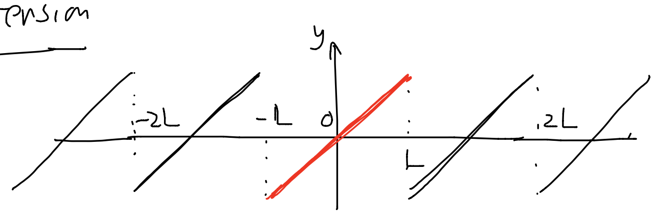
The result is an even function

symmetric with respect to the y-axis:
$F^e(-x) = F^e(x)$

"even extension of f" $\left\{ \begin{array}{l} \bullet\ F^e \text{ is } 2L\text{-periodic} \\ \bullet\ F^e(t) = \begin{cases} f(t) & 0<t<L \\ f(-t) & -L<t<0 \end{cases} \end{array} \right.$.

$F^e(-x) = F^e(x)$

· Odd Extension



period = 2

an odd function

symmetric with respect to the origin

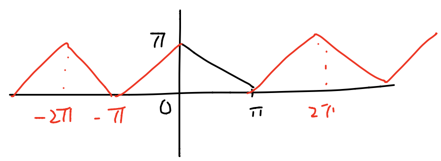
symmetric with resp. to the origin. $\boxed{F^o(-x) = -F^o(x)}$

"odd extension of f" $\begin{cases} \cdot\ F^o \text{ is } 2L\text{-periodic} \\ \cdot\ F^o(t) = \begin{cases} f(t) & 0<t<L \\ -f(-t) & -L<t<0 \end{cases} \end{cases}$.

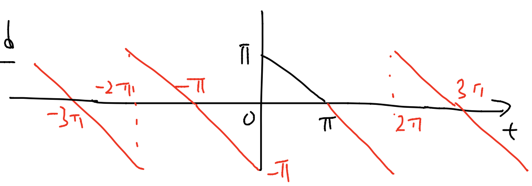
<u>Ex</u> Sketch even/odd extension of $f(x) = \pi - x \quad 0<x<\pi$.

<sol>

<u>even</u>



<u>odd</u>



☒

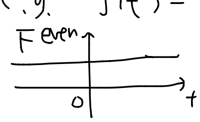
$f(t)$ on $0<t<L$ $\xrightarrow{\text{2L-periodic extension}}$

- even $\longrightarrow$ cosine series
- odd $\longrightarrow$ sine series

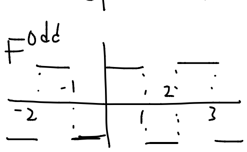
Fourier series

e.g. $f(t) = 1 \quad 0<t<\pi$

$a_0 = 2 \qquad a_k = 0$ for $k>0$



$\Rightarrow$ cosine series $f(t) = 1$.



$\Rightarrow$ sine series $f(t) = \displaystyle\sum_{k>0,\text{odd}} \frac{4}{k\pi}\sin(kt)$ ☒

$\boxed{\text{on } 0<t<\pi}$

Ex Compute Fourier cosine series for

$$f(t) = t \quad 0<t<1.$$

<sol> In principle,

$f(t) \longrightarrow$ even extension $\longrightarrow$ Fourier series $\longrightarrow$ get cosine series

In practice, can use

on $0<t<1$, $\quad t = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k - 1}{k^2\pi^2} \cdot 2 \cdot \cos(k\pi t)$

since

- for $k \neq 0$, $\quad a_k = \frac{2}{1}\int_0^1 t \cos(k\pi t)\,dt \underset{\text{exercise}}{=} 2 \cdot \frac{(-1)^k - 1}{k^2\pi^2}$.

this identity holds only on $0<t<1$.

- $a_0 = 2\int_0^1 t\,dt = 1$. ▨

On $-1<t<1$

The left hand side is $|t|$.