Lec 7

- Orthogonality
  - reason for the formula of Fourier coeff. c_k
  - Parseval's relation.

Next Monday

- even/odd extensions
  - Fourier cosine series
  - Fourier sine series

- **Orthogonality**: $L>0$.

For k, ℓ integer,

$$\boxed{\frac{1}{2L}\int_{-L}^{L} e^{ik\frac{\pi}{L}t} e^{-i\ell\frac{\pi}{L}t}\,dt = \begin{cases} 1 & \text{if } k=\ell \\ 0 & \text{if } k\neq\ell \end{cases}}$$

Rmk same result works for $\frac{1}{2L}\int_a^b e^{ik\frac{\pi}{L}t} e^{-i\ell\frac{\pi}{L}t}\,dt$

as long as $b-a = 2L =$ period of both $e^{ik\frac{\pi}{L}t}$ & $e^{i\ell\frac{\pi}{L}t}$!

Ex $f(t) = 3e^{i2t} + 4e^{-it} + 5e^{it}$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-i2t}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} 3e^{i2t} e^{-i2t}\,dt$$

$$+ \cancel{\frac{1}{2\pi}\int_{-\pi}^{\pi} 4e^{-it} e^{-i2t}\,dt} \to 0 \quad \text{by orthogonality}$$

$$+ \cancel{\frac{1}{2\pi}\int_{-\pi}^{\pi} 5e^{it} e^{-i2t}\,dt} \to 0$$

$$= 3\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i2t} e^{-i2t}\,dt$$

$$= 3\cdot 1 = 3 .$$

▯

Reason for formula of Fourier coeff. C_k "k-th Fourier coeff."

For $f(t)=\sum_{k=-\infty}^{\infty} C_k e^{ik\frac{\pi}{L}t}$

For fixed integer l,

$$\frac{1}{2L}\int_{-L}^{L} f(t)\, e^{-il\frac{\pi}{L}t}dt = \frac{1}{2L}\int_{-L}^{L}\sum_{k=-\infty}^{\infty} C_k e^{ik\frac{\pi}{L}t} e^{-il\frac{\pi}{L}t}dt$$

$$= \sum_{k=-\infty}^{\infty} C_k \cdot \underbrace{\frac{1}{2L}\int_{-L}^{L} e^{ik\frac{\pi}{L}t}\cdot e^{-il\frac{\pi}{L}t}dt}_{=\begin{cases}0 & \text{if } k\neq l\\ 1 & \text{if } k=l.\end{cases}}$$

$$= C_l$$

This calculation explains why we have such a formula for C_k in the Fourier series.
Fourier coefficient.

Parseval's relation

Ex Let $f(t)=e^{i\frac{t}{2}}+(2+i)e^{i2t}+\frac{1}{3}e^{i3t}$

- period? $T = 4\pi$ ← This is common period for $e^{i\frac{t}{2}}, e^{i2t}, e^{i3t}$

energy

$$\frac{1}{4\pi}\int_{-2\pi}^{2\pi}|f(t)|^2 dt = ?$$

(sol)

$$\frac{1}{4\pi}\int_{-2\pi}^{2\pi}|f(t)|^2 = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} f(t)\,\overline{f(t)}\,dt$$

Important! A complex number $\Rightarrow |A|^2 = A\bar{A}$, where $\bar{A}$ = complex conjugate.

$$= \frac{1}{4\pi}\int_{-2\pi}^{2\pi} \left(e^{i\frac{t}{2}} + (2+i)e^{i2t} + \frac{1}{3}e^{i3t}\right)\left(e^{-i\frac{t}{2}} + (2-i)e^{-i2t} + \frac{1}{3}e^{-i3t}\right)$$

$$= \frac{1}{4\pi}\int_{-2\pi}^{2\pi} e^{i\frac{t}{2}}\left(e^{-i\frac{t}{2}} + (2-i)e^{-i2t} + \frac{1}{3}e^{-i3t}\right)$$
$$+ \frac{1}{4\pi}\int_{-2\pi}^{2\pi} (2+i)\, e^{i2t}\left(\quad // \quad\quad // \quad \right)$$
$$+ \frac{1}{4\pi}\int_{-2\pi}^{2\pi} \frac{1}{3}e^{i3t}\left(\quad // \quad\quad // \quad \right)$$

(orthogonality gives cancelations!)

$$= 1 + (2+i)(2-i) + \left(\frac{1}{3}\right)^2$$
$$= 1 + |2+i|^2 + \left(\frac{1}{3}\right)^2 \quad \boxtimes$$

In general, for $f(t) = \sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi}{L}t}$ ← 2L-periodic

<u>Parseval relation</u>

$$\underbrace{\frac{1}{2L}\int_{-L}^{L} |f(t)|^2 dt}_{\text{"energy" of signal } f(t)} = \sum_{k=-\infty}^{\infty} |c_k|^2$$

← This measures the strength of the signal.

<u>Why this can be practical?</u>

We can <u>approximate</u> a (complicated) signal

by a <u>finite</u> sum of <u>simple</u> signals

by <u>"truncating"</u> the Fourier series

$f(t)$ ← original 2L-periodic signal → compute $c_k = \frac{1}{2L}\int_{-L}^{L} f(t)\, e^{-i\frac{k\pi}{L}t}\,dt$ for $k = 0, \pm 1, \pm 2, \cdots, \pm N$ → Use $f_N(t) = \sum_{k=-N}^{N} c_k e^{i\frac{k\pi}{L}t}$ to approximate $f(t)$

N ← some large number (A computer can handle it)

"here we loose the other terms $\sum_{|k|>N} c_k e^{i\frac{k\pi}{L}t}$

But, for large enough N, the size of error $\sum_{|k|>N} |c_k|^2$ will be small."

Also, we can estimate how much two signals are close when we know ONLY about Fourier coefficients

EX Suppose $f(t)$, $g(t)$ 2-periodic functions.

Let c_k = Fourier coeff. for f

d_k = 〃 〃 〃 g

Suppose we know that $|c_k - d_k| \le \frac{1}{100} \cdot 2^{-|k|}$.

Estimate $\frac{1}{2}\int_{-1}^{1} |f-g|^2\, dt$

<sol>. Note: Fourier coeff. of $f-g$

$= c_k - d_k$. ← show this! EASY.

Parseval ⟹ $\frac{1}{2}\int_{-1}^{1} |f-g|^2\,dt = \sum_{k=-\infty}^{\infty} |c_k - d_k|^2 \underset{\text{given condition}}{\le} \sum_{k=-\infty}^{\infty} \frac{1}{100^2}\, 2^{-2|k|}$

$$\leq \frac{1}{100^2} \sum_{k=-\infty}^{\infty} 2^{-2|k|}$$

$$= \frac{1}{100^2}\left[2^0 + 2\sum_{k=1}^{\infty} 2^{-2k}\right]$$

$$\sum_{k=1}^{\infty}\left(\frac{1}{2^2}\right)^k = \frac{\frac{1}{2^2}}{1-\frac{1}{2^2}} = \frac{1}{3}.$$

$$= \frac{1}{100^2}\left[1 + 2\cdot\frac{1}{3}\right].$$

$$= \frac{5}{30000} = \frac{1}{6000}$$

$$\therefore \frac{1}{2}\int_{-1}^{1} |f-g|^2\,dt \leq \frac{1}{6000}.$$

$\blacksquare$