Lec 6

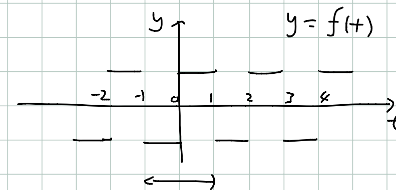
- Fourier Series – theory
  - period
  - piecewise differentiable functions
  - complex Fourier series

• period

$f(t)$ has period $T > 0$

$\underset{\text{def}}{\iff} \quad f(t+T) = f(t)$ for any $t \in \mathbb{R}$

e.g.



this pattern repeats
period = 2

e.g. $\sin t$, $\cos t$, e^{it} have period 2π.

check: $e^{i(t+2\pi)} = e^{it} \cdot \underbrace{e^{i2\pi}}_{1} = e^{it}$

e.g.
$L > 0$, $\sin(\frac{\pi}{L}t)$, $\cos(\frac{\pi}{L}t)$, $e^{i\frac{\pi}{L}t}$ have period $2L$.

e.g. $f(t)$, $g(t)$ period $T \implies f(t)g(t)$, $f(t) \pm g(t)$ have period T

e.g. n integer. $\sin(\frac{n\pi}{L}t)$, $\cos(\frac{n\pi}{L}t)$, $e^{i\frac{n\pi}{L}t}$ have period $\underline{2L}$.

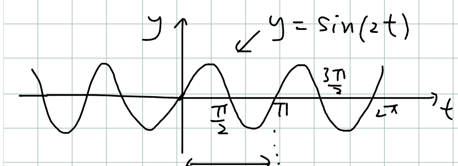
EX
- $\sum_{k=1}^{\infty} \beta_k \sin(\frac{k\pi}{L}t)$ has period $2L$.
- $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ has period 2π.

Rmk :

Fundamental period = smallest possible period

e.g. $\sin(2t)$ has period 2π

but also period π . ← fundamental period $= \pi$.



this pattern repeats.

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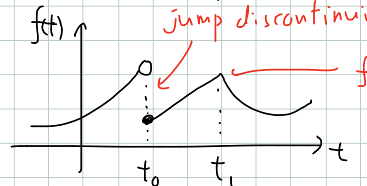
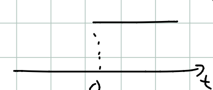
* piecewise differentiable functions

(i.e. "piecewise continuous functions with piecewise continuous derivatives)

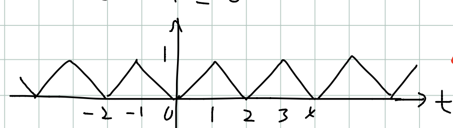


e.g. $f(t) = e^t$ differentiable (for any t), so piecewise differentiable.

e.g. $f(t) = \begin{cases} 1 & t > 0 \\ 0 & t \le 0 \end{cases}$



e.g. $f(t)$:



This is a 2-periodic function with $f(t) = \begin{cases} t & 0 \le t \le 1 \\ -t & -1 \le t \le 0 \end{cases}$.

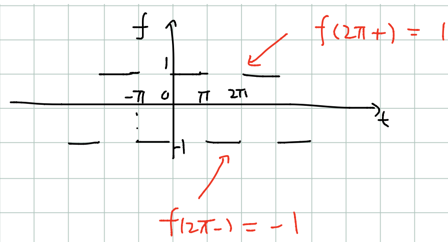
<u>notation</u>

$$f(t_0-) = \lim_{\substack{t \to t_0 \\ t < t_0}} f(t) \quad \text{"left limit"}$$

$$f(t_0+) = \lim_{\substack{t \to t_0 \\ t > t_0}} f(t) \quad \text{"right limit"}$$

<u>Ex 1</u> $f(t)$ 2π-periodic such that $f(t) = \begin{cases} 1 & 0 \le t < \pi \\ -1 & -\pi \le t < 0 \end{cases}$

$f(2\pi+)$, $f(2\pi-)$?



$\square$

(Complex) Fourier Series

Fourier series of 2π-periodic, piecewise differentiable $f(t)$:

$$\sum_{k=-\infty}^{\infty} c_k e^{ikt} \quad \left(\text{i.e.} = \cdots c_{-2}e^{-i2t} + c_{-1}e^{-it} + c_0 + c_1 e^{it} + c_2 e^{it} + \cdots\right)$$

where $c_k = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt$

More generally,

For f 2L-periodic

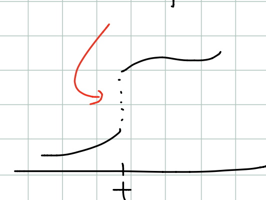
it is $\boxed{\displaystyle\sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi}{L}t}}$, $\boxed{c_k = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(t)\, e^{-i\frac{k\pi}{L}t}\, dt}$

Fourier series itself is a function of t!

Thm For 2L-periodic f,

"for most points t"

$$\sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi}{L}t} = \begin{cases} f(t) & \text{if } f \text{ has no jump at } t \\ \dfrac{f(t+)+f(t-)}{2} & \text{if } f \text{ has jump at } t \end{cases}$$



see notes p3, Example 2

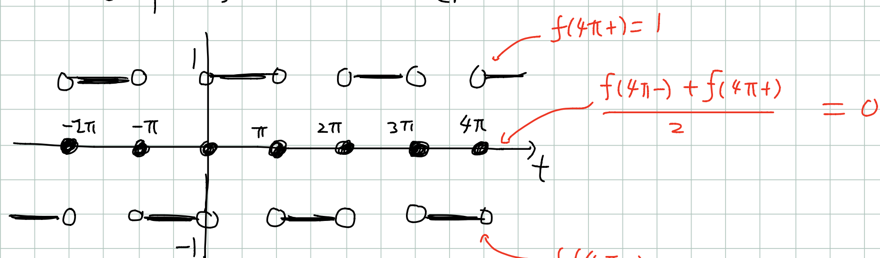
Ex 2 $f(t)$ 2π-periodic, $f(t) = \begin{cases} 1 & \text{if } 0 \le t < \pi \\ -1 & \text{if } -\pi \le t < 0 \end{cases}$

Then $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-ikt}$

$$= \begin{cases} -\frac{2}{k\pi} i, & k = \text{odd} \\ 0 & k = \text{even} \end{cases}$$

$\therefore$ Fourier series $\displaystyle\sum_{k=\text{odd}} \frac{-2}{k\pi} i \, e^{ikt}$

The graph of Fourier series



Compare this with the graph of $f(t)$: where do they differ?

$\blacksquare$

Rmk Why Fourier series is good for EE.

- can decompose signals according to "frequencies" (channels)

$f(t) =$ "many signals received/sent simultaneously" $c_k =$ "the signal at frequency k"

e.g. from many TV stations e.g. single TV channel.

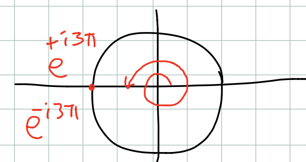
Reason for formula of c_k

* "orthogonality" (average correlation between different frequencies)

e.g. $\int_{-\pi}^{\pi} e^{i2t} e^{-i5t}\, dt = \int_{-\pi}^{\pi} e^{i(2-5)t}\, dt$

$= \int_{-\pi}^{\pi} e^{-i3t}\, dt = \frac{1}{-3i}\left[e^{-i3t}\right]_{-\pi}^{\pi} = \frac{1}{-3i}\left[\underbrace{e^{-i3\pi}}_{-1} - \underbrace{e^{+i3\pi}}_{-1}\right]$

$\frac{d}{dt} e^{ct} = c e^{ct}$



$= 0$

Similarly,

k, l integer

Important exercise

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt} e^{-ilt}\, dt = \begin{cases} 0 & \text{if } k \neq l \\ 1 & \text{if } k = l \end{cases}$$

- For 2L-periodic case $\frac{1}{2L}\int_{-L}^{L} e^{ik\frac{\pi}{L}t} e^{-il\frac{\pi}{L}t}\, dt = \begin{cases} 0, & k \neq l \\ 1, & k = l \end{cases}$

Ex $f(t) = e^{it} + e^{-i2t}$, $g(t) = e^{i2t} + e^{it}$

Compute $\int_{-\pi}^{\pi} f(t)\,\overline{g(t)}$

$\overline{g(t)}$ is the complex conjugate of $g(t)$

<sol> $$\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt = \int_{-\pi}^{\pi} \underbrace{(e^{it}+e^{-i2t})}_{f(t)} \underbrace{(e^{-i2t} + e^{-it})}_{\overline{g(t)}}$$

$$= \int_{-\pi}^{\pi} \left[e^{it}\cdot e^{-i2t} + e^{it}\cdot e^{-it} + e^{-i2t}\cdot e^{-i2t} + e^{-i2t}\cdot e^{-it} \right] dt$$

$$= \underbrace{\cancel{\int_{-\pi}^{\pi} e^{it} e^{-i2t}\,dt}}_{0} + \underbrace{\int_{-\pi}^{\pi} e^{it}\cdot e^{-it}\,dt}_{=2\pi} + \underbrace{\cancel{\int_{-\pi}^{\pi} e^{-i2t}\cdot e^{-i2t}\,dt}}_{0} + \underbrace{\cancel{\int_{-\pi}^{\pi} e^{-i2t}\cdot e^{-it}\,dt}}_{0}$$

$$= 2\pi$$ ▨