Lec 5

- Superposition & homogeneous conditions/eqns.
- inhomogeneous BC.
  - * steady states.

- homogeneous eqns/conditions

  : conditions that if functions u_1, u_2 satisfies them

  then for any constants c_1, c_2

  $c_1 u + c_2 u_2$ satisfies them, too!

  ← This property was the reason why we can do "superposition" of simple solutions u_k to make a general solution $\sum_{k=1}^{\infty} c_k u_k$ (satisfying PDE & BC) because we had homogeneous PDE & BC.

- Examples of homogeneous eqns/conditions.

  - homogeneous PDE's

  e.g.

  ① $u_t = c\, u_{xx}$, ② $u_{tt} = c^2 u_{xx}$

  - homogeneous BC:

  e.g.

  ① $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$, ② $\begin{cases} u_x(0,t) = 0 \\ u_x(L,t) = 0 \end{cases}$, ③ $\begin{cases} u(0,t) = u(L,t) \\ u_x(0,t) = u_x(L,t) \end{cases}$

  ④ $\begin{cases} u(0,t) = 0 \\ u_x(0,t) = 0 \end{cases}$, etc

← Note: So far we have considered only homogeneous case.

- Inhomogeneous conditions / eqns.

e.g. BC $\begin{cases} u(0,t) = 1 \\ u(L,t) = 0 \end{cases}$ (reason: Suppose u_1, u_2 satisfies the condition, then $(u_1+u_2)(0,t) = u_1(0,t) + u_2(0,t) = 1 + 1 = 2 \neq 1$. NOT satisfying it.)

e.g. $u_t = u_{xx} + 1$

(reason: $u_{1t} = u_{1xx} + 1$, $u_{2t} = u_{2xx} + 1$ $\Rightarrow$ $(u_1+u_2)_t = u_{1t} + u_{2t} = u_{1xx} + 1 + u_{2xx} + 1$
$= (u_1+u_2)_{xx} + 2$
So, not satisfying $u_t = u_{xx} + 1$.)

EX (Inhomogeneous BC)

Solve
$$\begin{cases} u_t = 3u_{xx} & 0 < x < 2;\ t > 0 \\ \boxed{\begin{matrix} u(0,t) = 6 \\ u(2,t) = -2 \end{matrix} \quad t > 0} \\ u(x,0) = f(x) = x, & 0 < x < 2 \end{cases}$$

Inhomogeneous BC.

Rmk Note, in Lec 4 we solved the same problem with different (homogeneous) BC.

Strategy: Reduce the problem to one with homogeneous BC

how? ① Find a particular sol. u_p to PDE & inhomogeneous BC $\begin{cases} u(0,t) = 6 \\ u(2,t) = -2 \end{cases}$
(Don't worry about IC here).

② Find general sol. u_h to the PDE & homogeneous BC
($u(0,t) = 0 = u(2,t)$)

③ Let $u(x,t) = u_h(x,t) + u_p$. ← This is a general sol. to PDE & inhomogeneous BC.

(check. $u(0,t) = u_h(0,t) + u_p(0,t) = 0 + 6 = 6$ ✓
$u(2,t) = u_h(0,t) + u_p(0,t) = 0 + (-2) = -2$ ✓)

(4) match the IC. (by finding appropriate constants c_k in $u_h = \sum_{k=1}^{\infty} c_k \sin(\frac{k\pi}{L}x) e^{-c(\frac{k\pi}{L})^2 t}$)

$$f(x) = u(x,0) = u_h(x,0) + \underbrace{u_p(x,0)}$$

since u_p is a particular sol. $u_p(x,0)$ is a particular function.

$$\Rightarrow f(x) - u_p(x,0) = u_h(x,0) = \sum_{k=1}^{\infty} c_k \sin(\frac{k\pi}{L}x).$$

apply Fourier sine series to determine c_k

i.e. $$c_k = \frac{2}{L}\int_0^2 \left(f(x) - u_p(x,0)\right) \sin(\frac{k\pi}{L}x)\, dx$$

- How to find u_p?

Try $u_p(x,t) = u_p(x)$ ← only one variable

- This particular sol. is called $\boxed{\text{steady state}}$ because it is time-indep

In heat eqn, $u(x,t) \xrightarrow[t\to\infty]{} $ steady state

<sol of EX>

(1) "steady state".

Find $u_p(x)$ solving

$$\begin{cases} u_t = 3u_{xx} \\ u(0,t) = 6 \\ u(2,t) = -2 \end{cases} \Rightarrow \begin{matrix} 0 = 3u_p'' \\ u_p(0) = 6 \\ u_p(2) = -2 \end{matrix} \Rightarrow \begin{matrix} u_p(x) = A + Bx \\ u_p(0) = A = 6 \\ u_p(2) = A + 2B = -2 \end{matrix}$$

$$\therefore \underbrace{u_p(x) = 6 - 4x}.$$

② "homogeneous case"

$$\begin{cases} u_t = 3u_{xx} \\ u(0,t)=0=u(2,t) \end{cases} \Longrightarrow \underset{\uparrow}{} \quad u_h = \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi}{2}x\right) e^{-3\left(\frac{k\pi}{2}\right)^2 t}$$

general sol. to homogeneous problem.

from last lecture

③ "general sol. to inhomogeneous case"

$$u(x,t) = u_p(x,t) + u_h(x,t)$$
$$= 6-4x + \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi}{2}x\right) e^{-3\left(\frac{k\pi}{2}\right)^2 t}.$$

④ Match IC.

$$x = f(x) = u(x,0) = 6-4x + \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi x}{2}\right).$$

$$\therefore \quad x-(6-4x) = \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi}{2}x\right).$$

For $k=1,2,3,\dots$

$$c_k = \frac{2}{2}\int_0^2 \left[x-(6-4x)\right]\sin\left(\frac{k\pi}{2}x\right)dx$$
$$= \int_0^2 (5x-6)\sin\left(\frac{k\pi}{2}x\right)dx$$
$$= 5\int_0^2 x\sin\left(\frac{k\pi}{2}x\right)dx - 6\int_0^2 \sin\left(\frac{k\pi}{2}x\right)dx$$

Here

- $$\int_0^2 \sin\left(\frac{k\pi}{2}x\right) = \left[-\frac{2}{k\pi}\cos\left(\frac{k\pi}{2}x\right)\right]_0^2 = -\frac{2}{k\pi}\left[\cos(k\pi)-1\right]$$
$$= -\frac{2}{k\pi}\left((-1)^k-1\right)$$

- $$\int_0^2 x\sin\left(\frac{k\pi}{2}x\right)dx = \frac{-4}{k\pi}\cos(k\pi) = \frac{-4}{k\pi}\cdot(-1)^k \qquad k=1,2,3,\dots$$

same calculation from last lecture.

So, $C_k = \frac{-6\cdot(-2)}{k\pi}\left((-1)^k - 1\right) + 5\cdot\frac{(-4)}{k\pi}(-1)^k$

$= \frac{1}{k\pi}\left[-8(-1)^k - 12\right]$

Finally,

$$u(x,t) = 6 - 4x + \sum_{k=1}^{\infty} \frac{1}{k\pi}\left[-8(-1)^k - 12\right] e^{-3\left(\frac{k\pi}{2}\right)^2 t}$$

$\blacksquare$