Lec. 4. - Wave eqn.: example
- Heat eqn: a full example.

Recall:

- "Fourier sine series" For $h(x)$ on $0<x<L$, Can write

$$\boxed{h(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}x\right) \quad \text{where } b_k = \frac{2}{L}\int_0^L h(x)\sin\left(\frac{k\pi}{L}x\right)dx, \quad k=1,2,3\cdots}$$

- "orthogonality" $n,k=1,2,3,4,\cdots$

$$\boxed{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)dx = \begin{cases} 0 & \text{if } n\neq k \\ \frac{L}{2} & \text{if } n=k \end{cases}}$$

Ex Solve
$$\begin{cases} u_{tt} = 4u_{xx} & \\ u(0,t) = 0 = u(1,t) & t>0 \\ u(x,0) = 1 & 0<x<1 \\ u_t(x,0) = \sin(2\pi x) - 2\sin(4\pi x) & \end{cases}$$

<sol>. $c=2$, $L=1$.

$c=2$, $L=1$

(WE) & (BC) $\Rightarrow$
$$u(x,t) = \sum_{k=1}^{\infty} \sin(k\pi x)\left[\alpha_k \cos(2k\pi t) + \beta_k \sin(2k\pi t)\right]$$

(IC) $\Rightarrow$
$$1 = u(x,0) = \sum_{k=1}^{\infty} \alpha_k \sin(k\pi x)$$

$$\sin(2\pi x) - 2\sin(4\pi x) = u_t(x,0) = \sum_{k=1}^{\infty} \beta_k \cdot 2k\pi \sin(k\pi x).$$

- $$\alpha_k = 2\int_0^1 1\,\sin(k\pi x)\,dx = 2\left[\frac{-\cos(k\pi x)}{k\pi}\right]_0^1 =$$

$L=1$ $\quad$ $h(x)$ $\quad$ $k=1,2,3,\cdots$

$$= \frac{2}{k\pi}\left[-\cos(k\pi) + 1\right] = \frac{2}{k\pi}\left[-(-1)^k + 1\right]$$

$$\therefore \quad \alpha_k = \begin{cases} 0 & k = \text{even} \\ \frac{2}{k\pi} & k = \text{odd} \end{cases}$$

- $$2k\pi\beta_k = 2\int_0^1 \left[\sin(2\pi x) - 2\sin(4\pi x)\right]\sin(k\pi x)\,dx$$

use "orthogonality"

$$= 2\left[\underbrace{\int_0^1 \sin(2\pi x)\sin(k\pi x)\,dx}_{= \begin{cases} 0 & k \neq 2 \\ \frac{1}{2} & k = 2 \end{cases}} - 2\underbrace{\int_0^1 \sin(4\pi x)\sin(k\pi x)\,dx}_{= \begin{cases} 0 & k \neq 4 \\ \frac{1}{2} & k = 4 \end{cases}}\right]$$

$$\therefore \quad 2k\pi\beta_k = \begin{cases} 1 & k = 2 \\ -2 & k = 4 \\ 0 & k \neq 2, 4 \end{cases}$$

$$\therefore \quad \beta_k = \begin{cases} \frac{1}{4\pi} & k = 2 \\ -\frac{1}{4\pi} & k = 4 \\ 0 & k \neq 2, 4 \end{cases}$$

Finally

$$\boxed{\begin{aligned} u(x,t) = &\sum_{\substack{k=1 \\ k=\text{odd}}}^{\infty} \frac{2}{k\pi}\sin(k\pi x)\cos(2k\pi t) \\ &+ \frac{1}{4\pi}\sin(2\pi x)\sin(4\pi t) \\ &- \frac{1}{4\pi}\sin(4\pi x)\sin(8\pi t). \end{aligned}}$$

- Summary for wave eqn.

  with BC : $\underline{u(0,t) = 0 = u(L,t)}$

Recall.

Let $c, L > 0$ given constants

$$\begin{cases} (WE) & u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \\ (BC) & \underline{u(0,t) = 0 = u(L,t)} \quad t > 0 \end{cases}$$

special form

- Plug in $u(x,t) = X(x)T(t)$ & solve two ODE's for $X(x)$ ($\leftarrow$ eigenvalue problem) & $T(t)$

$$\Rightarrow \text{get} \quad u_k(x,t) = \sin\left(\frac{k\pi}{L}x\right)\left[\alpha_k \cos\left(\frac{k\pi c}{L}t\right) + \beta_k \sin\left(\frac{k\pi c}{L}t\right)\right\}$$

$k = 1, 2, 3, \ldots$

- general sol.

$$\boxed{u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{L}x\right)\left\{\alpha_k \cos\left(\frac{k\pi c}{L}t\right) + \beta_k \sin\left(\frac{k\pi c}{L}t\right)\right\}}$$

α_k, β_k arbitrary constants !

Remark Different BC gives different solution.

e.g. If (BC) $u_x(0,t) = 0 = u_x(L,t)$, we get different solution,

$$u(x,t) = \alpha_0 + \sum_{k=1}^{\infty} \cos\left(\frac{k\pi}{L}x\right)\left\{\alpha_k \cos\left(\frac{k\pi c}{L}t\right) + \beta_k \sin\left(\frac{k\pi c}{L}t\right)\right\}$$

- Matching (IC) : $\begin{cases} u(x,0) = f(x), \\ u_t(x,0) = g(x) \end{cases} \quad 0 < x < L$

$$f(x) = u(x,0) = \sum_{k=1}^{\infty} \alpha_k \sin\left(\frac{k\pi}{L}x\right)$$

$$g(x) = u_t(x,0) = \sum_{k=1}^{\infty} \beta_k \frac{k\pi c}{L} \sin\left(\frac{k\pi}{L}x\right)$$

Use $h(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{L}\right)$

$$b_k = \frac{2}{L}\int_0^{2} h(x) \sin\left(\frac{k\pi}{L}x\right) dx$$

Heat eqn

$u(x,t)$ temparature at (x,t)
satisfies

$$u_t = c\, u_{xx}, \qquad c>0, \text{ constant depending on the media}$$

Rmk $u(x,t)$ can be - Concentration of a chemical
- probability for a random motion, etc

Ex Solve
$$\begin{cases} u_t = 3\, u_{xx} & 0<x<2,\ t>0 \\ u(0,t) = 0 = u(2,t) & t>0 \\ u(x,0) = f(x) = x, & 0<x<2. \end{cases}$$

Rmk $u_t = c\, u_{xx}$ has only one t-derivative, so only one BC.

<sol>

• Step 1 $X(x)T(t)$. $\quad XT' = 4X''T$

"Separation of variables"

$$\Longrightarrow \frac{T'}{3T} = \frac{X''}{X} = -\gamma$$

Use (BC).

$\boxed{1}$ $\begin{cases} X''(x) + \gamma X(x) = 0 \\ X(0) = 0 = X(2) \end{cases}$ $\leftarrow$ eigenvalue problem

$\boxed{2}$ $T'(t) + 3\gamma T(t) = 0$

$\boxed{1} \Longrightarrow$ $\quad \gamma = \left(\frac{k\pi}{2}\right)^2 \quad k=1,2,3,\dots$

Last lecture

$$\underline{X_k(x) = \sin\left(\frac{k\pi}{2}x\right)}$$

$\boxed{2} \Longrightarrow$ $\quad \underline{T_k(t) = e^{-3\left(\frac{k\pi}{2}\right)^2 t}}$

Step 2 "superposition"

general solution

$$u(x,t) = \sum_{k=1}^{\infty} \underbrace{C_k}_{} \sin\left(\frac{k\pi}{2}x\right) e^{-3\left(\frac{k\pi}{2}\right)^2 t}$$

C_k : arbitrary constant.

Step 3 "match IC"

$$f(x) = u(x,0) = \sum_{k=1}^{\infty} C_k \sin\left(\frac{k\pi}{2}x\right)$$

$L=2$

$$\therefore\ C_k = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{k\pi}{2}x\right)dx$$

$L=2$

$L=2$

Note $f(x) = x$.

$$C_k = \int_0^2 x\sin\left(\frac{k\pi}{2}x\right)dx$$

$$= \left[x\cdot\frac{-2}{k\pi}\cos\left(\frac{k\pi}{2}x\right)\right]_0^2 - \int_0^2 1\cdot\frac{-2}{k\pi}\cos\left(\frac{k\pi}{2}\right)dx$$

Integration by parts

$$\boxed{\begin{aligned}&\int g h' \\ &= gh - \int g' h\end{aligned}}$$

note : $k = 1, 2, 3 \ldots$

$\therefore k \neq 0.$

$$= 2\cdot\frac{-2}{k\pi}\cos(k\pi) - \left[\frac{-2}{k\pi}\cdot\frac{2}{k\pi}\sin\left(\frac{k\pi}{2}\right)\right]_0^2$$

$$= -\frac{4}{k\pi}\cos(k\pi) - \frac{-2}{k\pi}\cdot\frac{2}{k\pi}\left[\underset{0}{\cancel{\sin(k\pi)}} - \underset{0}{\cancel{\sin(0)}}\right]$$

$\therefore \quad C_k = \frac{-4}{k\pi}\cos(k\pi) = \underbrace{\frac{-4}{k\pi}\cdot(-1)^k}$ $\quad k=1,2,3,\cdots$

Finally

$$u(x,t) = \sum_{k=1}^{\infty} \frac{-4\cdot(-1)^k}{k\pi}\sin\left(\frac{k\pi}{2}x\right) e^{-3\cdot\left(\frac{k\pi}{2}\right)^2 t}$$

$\boxed{///}$

<u>Rmk</u> Note $e^{-3\left(\frac{k\pi}{2}\right)^2 t} \longrightarrow 0$ as $t \to +\infty$.

So $u(x,t) \longrightarrow 0$ as $t \to +\infty$.

i.e. as time goes on the temperature converge to 0, in this example.