Lec 23

- Announcements
  - final exam : info to be sent.
  - pass ? 64% in the final exam.
  - office hours (TBA) ; also check CONNECT.
  - course evaluation
- Some practice with LTI & z-transform.

EX* $y[n] - 5y[n-1] + 6y[n-2] = 2x[n] - 5x[n-1]$.

Assume: causal.

(a) Find $H(z)$. ($H(z)$ is z-transform of $h[n]$)

"system function" — impulse response function

(b) Find $h[n]$

(c) Find $y[n]$ if $x[n] = \delta[n-4]$

(d) Find $y[n]$ if $x[n] = u[n]$.

<sol>

(a) Z-transform

$$Y(z) - 5z^{-1}Y(z) + 6z^{-2}Y(z) = 2X(z) - 5z^{-1}X(z)$$

time-shifting property $\quad y[n-n_0] \xrightarrow{Z} z^{-n_0}Y(z)$

$$(1 - 5z^{-1} + 6z^{-2})\,Y(z) = (2 - 5z^{-1})\,X(z)$$

$$Y(z) = \underbrace{\frac{2 - 5z^{-1}}{1 - 5z^{-1} + 6z^{-2}}}_{H(z)}\,X(z)$$

(When $x[n] = \delta[n]$, (i.e. when $X(z) = 1$) corresponding output is $y[n] = H[n]$.) So, $Y(z) = H(z)$

$$\therefore\ H(z) = \frac{2 - 5z^{-1}}{1 - 5z^{-1} + 6z^{-2}} = \frac{2z^2 - 5z}{z^2 - 5z + 6} \qquad \blacksquare$$

(b) First, partial fractions

$$H(z) = \frac{z(2z-5)}{z^2-5z+6} = z \cdot \frac{(2z-5)}{(z-3)(z-2)} = z \cdot \left[\frac{A}{z-3} + \frac{B}{z-2}\right]$$

$$A(z-2) + B(z-3) = 2z-5$$

$$(A+B)z - 2A - 3B = 2z - 5$$

$$A+B = 2, \quad -2A-3B = -5$$

$$\therefore A = 1, \; B = 1.$$

$$\therefore H(z) = z\left(\frac{1}{z-3} + \frac{1}{z-2}\right)$$

$$= \frac{z}{z-3} + \frac{z}{z-2}$$

$z=3$, $z=2$ are poles.

To get inverse z-transform, NEED TO determine ROC.

NOTE: poles at $z=3$, $z=2$.

<u>possible ROC's</u>:
- $|z|>3$ ⟵ choose this because <u>causality</u>
- ~~$2<|z|<3$~~
- ~~$|z|<2$~~

$\Longleftrightarrow$ • $h[n]=0 \;\; \forall n<0$

$\Longleftrightarrow$ • ROC for $H(z)$: $\boxed{|z|>|a|}$

• $\lim_{|z|\to+\infty} H(z)$ exists.

ROC for $H(z)$: $|z|>3$.

$\therefore$ $\frac{z}{z-3}$; ROC: $|z|>3$

$\frac{z}{z-2}$; ROC $|z|>2$

This is because the other possible ROC $|z|<2$ intersected with $|z|>3$. does not give the ROC of $H(z)$ $|z|>3$.

<u>Tip</u> For a <u>causal</u> system, for each term in $H(z)$ after partial fraction has ROC of the form $|z|>|a|$.

$\therefore$

$$H(z) \xrightarrow{\mathcal{Z}^{-1}} \underbrace{h[n] = (3^n + 2^n)u[n]}$$

$a^n u[n] \xrightarrow{\mathcal{Z}} \frac{z}{z-a}$; ROC $|z|>|a|$

$\boxed{//}$

(c)

Note: $y[n] = (h * x)[n]$

$$\downarrow \mathcal{Z}$$

$$Y(z) = H(z)X(z)$$

$x[n] = \delta[n-4] = \delta_4[n]$ ← notation

$$y[n] = (h * \delta_4)[n] = h[n-4] = \underbrace{(3^{n-4} + 2^{n-4})u[n-4]}$$

$$= \sum_{m=-\infty}^{\infty} h[m]\delta_4[n-m]$$

$$= \sum_{m=-\infty}^{\infty} h[m]\delta[n-m-4] = h[n-4]$$

← nonzero only when $n-m-4=0$

$\blacksquare$

(d) $x[n] = u[n]$

$$y[n] = (h * x)[n] = 3^n u[n] * u[n] + 2^n u[n] * u[n]$$

$$= \sum_{m=-\infty}^{\infty} (3^m + 2^m)\, u[m]\underbrace{u[n-m]}$$

← nonzero for $n-m \geq 0$ i.e. $m \leq n$.

$$= \sum_{m=-\infty}^{n} (3^m + 2^m)\underbrace{u[m]}$$

← nonzero for $m \geq 0$

$$= \begin{cases} 0 & n < 0 \\ \sum_{m=0}^{n} (3^m + 2^m) & n \geq 0 \end{cases}$$

$$= u[n] \sum_{m=0}^{n} (3^m + 2^m)$$

$$= u[n] \cdot \left(\frac{1-3^{n+1}}{1-3} + \frac{1-2^{n+1}}{1-2} \right) = \underbrace{u[n] \cdot \left(\frac{3^{n+1}-1}{2} + 2^{n+1} - 1 \right)}$$

Alternative method

$$x[n] = u[n] \xrightarrow{\mathcal{Z}} X(z) = \frac{z}{z-1}$$

ROC: $|z| > 1$.

$$Y(z) = H(z) X(z)$$

$$= \underbrace{\left(\frac{z}{z-3} + \frac{z}{z-2} \right)} \frac{z}{z-1}$$

partial fraction for $H(z)$

$$= \frac{z^2}{(z-3)(z-1)} + \frac{z^2}{(z-2)(z-1)}$$

$$= \frac{z^2}{2} \left[\frac{1}{z-3} - \frac{1}{z-1} \right] + z^2 \left[\frac{1}{z-2} - \frac{1}{z-1} \right]$$

Will make time-shift

will make time shift

$$= \frac{z}{2} \left[\frac{z}{z-3} - \frac{z}{z-1} \right] + z \left[\frac{z}{z-2} - \frac{z}{z-1} \right]$$

To use inverse z-transform of $\frac{z}{z-a}$

ROC $|z| > 3$ ROC $|z| > 1$ ROC $|z| > 2$ ROC $|z| > 1$

☆ We chose these ROCs so that their intersection is ROC of $H(z)X(z)$, that is intersection of $|z| > 3$ & $|z| > 1$

Thus,

$$Y(z) \xrightarrow{Z^{-1}} y[n] = \frac{1}{2}\left(3^{n+1}u[n+1] - u[n+1]\right) + 2^{n+1}u[n+1] - u[n+1]$$

time shift property.

$$= \underbrace{\left(\frac{3^{n+1}-1}{2} + 2^{n+1} - 1\right) u[n+1]}$$ ▨

Rmk

Check the two methods give the same results:

method 1 → $$\left(\frac{3^{n+1}-1}{2} + 2^{n+1} - 1\right) u[n] = \begin{cases} \frac{3^{n+1}-1}{2} + 2^{n+1} - 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

method 2 → $$\left(\frac{3^{n+1}-1}{2} + 2^{n+1} - 1\right) u[n+1] = \begin{cases} \frac{3^{n+1}-1}{2} + 2^{n+1} - 1 & \text{for } n \geq -1 \\ 0 & \text{for } n < -1 \end{cases}$$

at $n=-1$,

$$\frac{3^{-1+1}-1}{2} + 2^{-1+1} - 1 = 0.$$

$$= \begin{cases} \frac{3^{n+1}-1}{2} + 2^{n+1} - 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

So, they are the same results.