Lec 22

* Properties of z-transform.
  - time shift
  - convolution.
* Causality

* Properties of z-transform.

※ similar to F.T. but, be careful with ROC.

Time-shift

$$\boxed{f[n-n_0] \xrightarrow{Z} z^{-n_0} \underbrace{F(z)}_{\text{z-transform of } f[n]} \quad \text{with same ROC}}$$

Ex $h[n] = a^n u[n-n_0]$

<sol> $h[n] = a^{n_0} \underbrace{a^{n-n_0} u[n-n_0]} \xrightarrow{Z} a^{n_0} \underbrace{z^{-n_0} \frac{z}{z-a}}$; ROC $|z|>|a|$

using time-shift

& $a^n u[n] \xrightarrow{Z} \frac{z}{z-a}$; ROC $|z|>|a|$ ▨

Ex Suppose $H(z) = \frac{1}{z-a}$ with ROC $|z|>|a|$.

Find $h[n]$.

<sol> $H(z) = z^{-1} \frac{z}{z-a}$; ROC : $|z|>|a|$

$\xrightarrow{Z} \underbrace{a^{n-1} u[n-1]}$ time shift ▨

* Convolution

$$\boxed{(f*g)[n] \xrightarrow{\mathcal{Z}} F(z)\cdot G(z) \qquad ROC = (\text{ROC of } F(z)) \cap (\text{ROC of } G(z))}$$

EX $f[n] = \left(\frac{1}{2}\right)^n u[n]$

$g[n] = -2^n u[-n-1]$

$(f*g)[n] = ?$

<sol) **Method 1** Direct computation of

$$(f*g)[n] = \sum_{m=-\infty}^{\infty} f[m]\, g[n-m]$$

Method 2 Use $\mathcal{Z}$-transform.

$$(f*g)[n] \xrightarrow{\mathcal{Z}} F(z)\,G(z)$$

$$f[n] = \left(\frac{1}{2}\right)^n u[n] \xrightarrow{\mathcal{Z}} \frac{z}{z-\frac{1}{2}} \qquad ROC \ |z| > \frac{1}{2}$$

$$g[n] = -2^n u[-n-1] \xrightarrow{\mathcal{Z}} \frac{z}{z-2} \ ; ROC\ |z| < 2$$

$$F(z)\,G(z) = \frac{z^2}{(z-\frac{1}{2})(z-2)} \qquad ROC \ \underline{\tfrac{1}{2} < |z| < 2}$$

intersection of $\frac{1}{2} < |z|$ & $|z| < 2$

Partial fraction

$$= z^2 \cdot \frac{1}{(z-\frac{1}{2})(z-2)}$$

$$= z^2 \cdot \left(-\frac{2}{3}\right)\left(\frac{1}{z-\frac{1}{2}} - \frac{1}{z-2}\right)$$

$$= -\frac{2}{3} \cdot z \left(\frac{z}{z-\frac{1}{2}} - \frac{z}{z-2}\right)$$

Now, $\frac{z}{z-\frac{1}{2}}$; $\boxed{\text{ROC } |z| > \frac{1}{2}} \xrightarrow{\mathcal{Z}^{-1}} \left(\frac{1}{2}\right)^n u[n]$

$\frac{z}{z-2}$; $\boxed{\text{ROC } |z| < 2} \xrightarrow{\mathcal{Z}^{-1}} -2^n u[-n-1]$

we chose these ROC's so that their intersection is $\frac{1}{2} < |z| < 2$

$$\therefore F(z)G(z) = -\frac{2}{3} z \left(\frac{z}{z-\frac{1}{2}} - \frac{z}{z-2}\right)$$

$$\xrightarrow{\mathcal{Z}^{-1}} (f * g)[n]$$

time-shift

$$= -\frac{2}{3} \cdot \left(\left(\frac{1}{2}\right)^{n+1} u[n+1] - \left(-2^{n+1} u[-(n+1)-1]\right)\right)$$

$$= -\frac{2}{3}\left(\left(\frac{1}{2}\right)^{n+1} u[n+1] + 2^{n+1} u[-n-2]\right).$$

* **Causality** "present depends only on the past/present"
(e.g. motion of automobile)

An LTI system is <u>causal</u> if output $y[n]$ depends <u>only</u> on $\begin{cases}\text{present} \\ \text{past}\end{cases}$ inputs.
(e.g. $x[n], x[n-1], x[n-2], \cdots$)

e.g. · $y[n] = x[n] - x[n-1]$ ← causal

· $y[n] = x[n] - x[n+1]$ ← <u>NOT</u> causal.

The following are equivalent

① LTI system is causal

$\Updownarrow$

② $h[n] = 0$ for $n < 0$ — impulse response function

$\Updownarrow$

③ $H(z)$ has ROC <u>outside</u> a disk (i.e. $\{|z| > R\}$)
<u>and</u> $\lim_{|z| \to +\infty} H(z)$ exists

Some explanations

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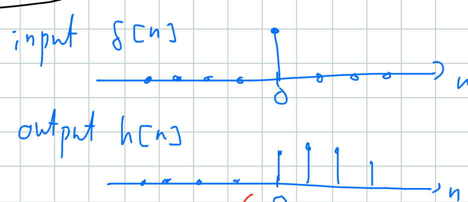


before $n=0$ there was no (zero) input, so causality implies no (zero) output before $n=0$.

- For LTI system, for input $x[n]$

$$\underbrace{y[n]}_{\text{output}} = (h * x)[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

causality $\Rightarrow$ no contribution from $x[n+1], x[n+2], \dots$ to $y[n]$

Thus $h[m] = 0$ for $m < 0$.

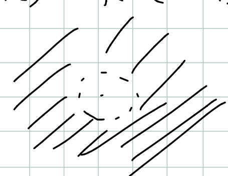
- $$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=0}^{\infty} h[n] z^{-n}$$

If causal

For $n \geq 0$, the larger $|z|$, the smaller $|z^{-n}|$ so, the better the convergence of $\sum_{n=0}^{\infty} h[n] z^{-n}$.

So, once the series converges for a complex number z_0 (i.e. $z_0 \in ROC$) then for any z with $|z| > |z_0|$ the series converges, i.e. $z \in ROC$

Therefore, ROC is outside a disk:



because the ROC is the union of such sets of type $\{|z| > r\}$

(e.g. $\{|z|>1\} \cup \{|z|>2\} = \{|z|>1\}$.

e.g. $h[n] = 2^n u[n]$ causal ($\because$ for $n<0$ $h[n] = 0$)

Here, $H(z) = \frac{z}{z-2}$, ROC $|z|>2$. & $\lim_{|z|\to+\infty} H(z) = 1$

e.g. $h[n] = -2^n u[-n-1]$ not causal $H(z) = \frac{z}{z-2}$; ROC $|z|<2$

(e.g. $h[-1] = -2^{-1} \neq 0$)

EX Suppose $H(z) = \frac{z}{z-1}$ for a causal LTI system

Find $h[n]$

<sol>

ROC either $|z|<1$ $\leftarrow$ NOT causal

or $|z|>1$ $\leftarrow$ choose this

$\therefore$ $\frac{z}{z-1}$; ROC $|z|>1$ $\xrightarrow[\text{use basic example}]{\mathcal{Z}^{-1}}$ $h[n] = 1^n u[n] = u[n]$ ▨

$a^n u[n] \xrightarrow{\mathcal{Z}} \frac{z}{z-a}$; ROC $|z|>|a|$

$-a^n u[-n-1] \longrightarrow \frac{z}{z-a}$; ROC $|z|<|a|$

lesson "Causality determines ROC".

EX* Suppose LTI system

$$\frac{1}{2} y[n] = y[n-1] + x[n]$$

is causal.

Let $x[n] = u[n]$

Find $y[n]$

<sol>. <u>rmk</u> $y[n] = (h * x)[n]$

- z-transform of LTI

$$\frac{1}{2} Y(z) = \overbrace{z^{-1} Y(z)}^{\text{time-shift}} + X(z)$$

$$\left(\frac{1}{2} - z^{-1}\right) Y(z) = X(z)$$

$$\therefore \quad Y(z) = \underbrace{\boxed{\frac{1}{\frac{1}{2} - z^{-1}}}}_{H(z)} X(z)$$

- $H(z) = \dfrac{1}{\frac{1}{2} - z^{-1}}$ ← z-transform of $h[n]$

$$= \frac{2z}{z-2}$$ pole $z=2$.

Causality $\Longrightarrow$ ROC ; $\underline{|z| > 2}$.

outside a disk.

- $x[n] = u[n] \xrightarrow{Z} X(z) = \frac{z}{z-1}$; ROC $|z| > 1$

$$\therefore \quad Y(z) = H(z) X(z)$$

$$= \frac{2z}{z-2} \cdot \frac{z}{z-1}$$; ROC "$|z| > 1$" $\cap$ "$|z| > 2$"

$\therefore$ $\underline{\underline{|z| > 2}}$

To get $y[n]$,

$$Y(z) = 2z^2 \frac{1}{(z-2)(z-1)}$$ ← partial fractions

$$= 2z^2 \left(\frac{1}{z-2} - \frac{1}{z-1} \right)$$

$$= 2z \left(\frac{z}{z-2} - \frac{z}{z-1} \right) \quad \text{with ROC } |z| > 1$$

$$\frac{z}{z-2}\ ;\ \text{ROC } \boxed{|z|>2} \xrightarrow{\mathcal{Z}^{-1}} 2^n u[n]$$

$$\frac{z}{z-1}\ ;\ \text{ROC } \boxed{|z|>1} \xrightarrow{\mathcal{Z}^{-1}} u[n]$$

to have intersection $|z|>1$, we have to choose these ROC's (e.g. $|z|<2$ does not work for the intersection $|z|>2$)

$$\therefore\ Y(z) \xrightarrow{\mathcal{Z}^{-1}} y[n] = 2 \cdot \left(2^{n+1} u[n+1] - u[n+1] \right)$$

$$\parallel$$

$$2z\left(\frac{z}{z-2} - \frac{z}{z-1} \right)$$

time-shift.

$$\therefore\ y[n] = (2^{n+2} - 2)\, u[n+1].$$

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