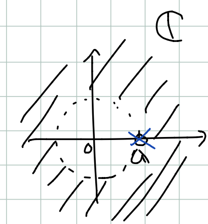
Lec 21

- z-transform (useful when F.T. does not exist / converge)
  - examples, ROC, inverse z-transform.
  - causality (If time permits)

$$h[n] \xrightarrow{\mathcal{Z}} H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} \quad \text{for } z \in ROC$$

(Last lecture)

Ex $h[n] = a^n u[n]$ $a \neq 0$ any fixed constant.

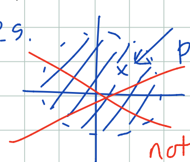


$$H(z) = \frac{z}{z-a} \; ; \; ROC : |z| > |a|$$

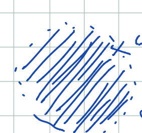
In this case $z=a$ is a pole : z_0 is a pole if $H(z_0)$ has zero in the denominator

" Poles are not in ROC. (only possibly on the boundary of ROC)

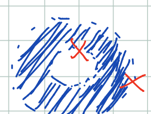
e.g.



e.g.



e.g.



Ex* $h[n] = -a^n u[-n-1]$ $a \neq 0$ fixed.

$H(z) = ?$

<sol> $H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$

$$= \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$-n-1 \geq 0 \iff n \leq -1$

$$= -\sum_{k=1}^{\infty} a^{-k} z^{k} \qquad k=-n$$

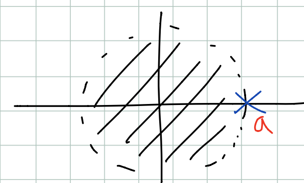
$$= -\sum_{k=1}^{\infty} \left(\frac{z}{a}\right)^{k} \qquad \sum_{k=1}^{\infty} r^{k} = \frac{r}{1-r} \quad \text{for } |r|<1$$

$$= -\frac{z}{a} \cdot \frac{1}{1-\frac{z}{a}} \qquad \text{for } \left|\frac{z}{a}\right| < 1$$

i.e. $|z| < |a|$

$$= \frac{z}{z-a}. \quad \text{with ROC } |z| < |a|.$$

$$\therefore \underline{H(z) = \frac{z}{z-a}; \ \text{ROC } |z| < |a|.}$$



Two basic examples

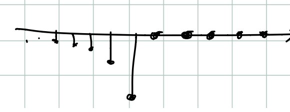
$a \neq 0$

$h[n]$	$H(z)$	ROC
$a^{n} u[n]$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
$-a^{n} u[-n-1]$	$\frac{z}{z-a}$	$\|z\| < \|a\|$

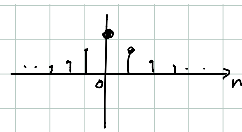
※ This example suggests that to recover $h[n]$ from $H(z)$, we need to know ROC!

Ex. $H(z) = \frac{z}{z-2}$; ROC $|z|>2$ $\xrightarrow{\mathcal{Z}^{-1}}$ $h[n] = 2^{n} u[n]$

$H(z) = \frac{z}{z-2}$; ROC $|z|<2$ $\xrightarrow{\mathcal{Z}^{-1}}$ $h[n] = -2^{n} u[-n-1]$



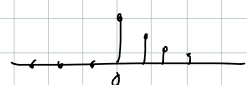
Ex* $h[n] = 2^{-|n|}$

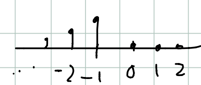


z-transform?

<sol> Can use basic example

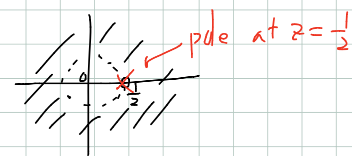
$$h[n] = 2^{-n} u[n] + 2^{n} u[-n-1]$$



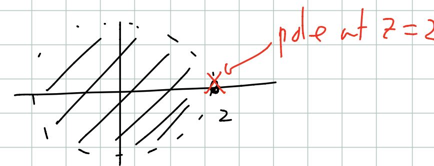


$$= 2^{-n} u[n] - (-2^{n} u[-n-1])$$

$2^{-n} u[n] \xrightarrow{\mathcal{Z}} \dfrac{z}{z - \frac{1}{2}}$; ROC : $\frac{1}{2} < |z|$



$-2^{n} u[n] \xrightarrow{\mathcal{Z}} \dfrac{z}{z-2}$; ROC : $|z| < 2$

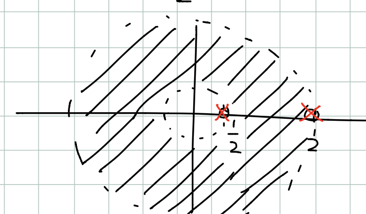


$\therefore\ 2^{-|n|} = (\frac{1}{2})^{n} u[n] - (-2^{n} u[-n-1])$

$$\xrightarrow{\mathcal{Z}} \frac{z}{z - \frac{1}{2}} - \frac{z}{z-2} \ ; \ \text{ROC} : \{\tfrac{1}{2} < |z|\} \cap \{|z| < 2\}$$

i.e. $\frac{1}{2} < |z| < 2$.

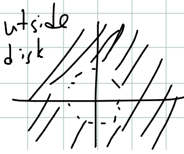
* When adding/multiplying z-transforms, intersect ROC's.

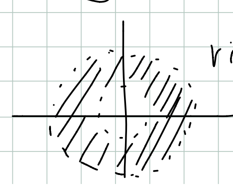


* ROC : - Never contain poles
  - Poles are possibly on the boundary
  - either

centered at origin

disk 

outside a disk 

 ring

Rmk For some signals, z-transform may not exist.

e.g. $h[n] = 2^n$

$\therefore$ $H(z) = \sum_{n=-\infty}^{\infty} 2^n z^{-n} = \sum_{n=0}^{\infty} 2^n z^{-n} + \sum_{n=-1}^{-\infty} 2^n z^{-n}$

$$= \sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n + \sum_{k=1}^{\infty}\left(\frac{z}{2}\right)^k \quad \leftarrow k=-n$$

converges for $\left|\frac{2}{z}\right| < 1$ $\qquad$ converges for $\left|\frac{z}{2}\right| < 1$

no intersection of these two ROCs

$\Rightarrow$ no z for which the above series for $H(z)$ converges.

Inverse z-transform

There is a formula for inverse z-transform.

(need to know "complex analysis") $\quad h[n] = \mathcal{Z}^{-1}\{H(z)\}[n] = \frac{1}{2\pi i}\oint_C H(z) z^{n-1}\, dz$

This is an integral along a closed curve around the origin, inside ROC

In this course, we use basic examples to determine some inverse z-transforms.

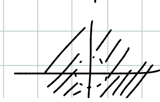
Ex* Let $H(z) = \frac{z}{z-\frac{1}{2}} - \frac{z}{z-2}$; ROC : $|z| > 2$

Find $h[n]$

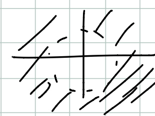
<sol> To do $\mathcal{Z}^{-1}$, need to know ROC

ROC for $\frac{z}{z-\frac{1}{2}}$: either

$\nearrow$ $|z| < \frac{1}{2}$  $\leftarrow$ Cannot choose this, why?

$\searrow$ $|z| > \frac{1}{2}$  $\leftarrow$ choose.

ROC for $\frac{z}{z-2}$: either $\rightarrow$ $|z|<2$  $\leftarrow$ cannot choose this, why?

$\rightarrow$ $|z|>2$  $\leftarrow$ choose

Require the intersection to be $|z|>2$.

Thus,
$$\begin{cases} \text{ROC for } \frac{z}{z-\frac{1}{2}} ; \ |z|>\frac{1}{2} \xrightarrow{\mathcal{Z}^{-1}} \left(\frac{1}{2}\right)^n u[n] \\ \text{ROC for } \frac{z}{z-2} : \ |z|>2 \xrightarrow{\mathcal{Z}^{-1}} 2^n u[n]. \end{cases}$$

$$\therefore \ h[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u[n]$$

EX Let $H(z) = \frac{1}{z^2+2z-8}$; ROC: $2<|z|<4$

$h[n] = ?$

<sol> $H(z) = \frac{1}{(z+4)(z-2)}$

$$= \frac{z}{3}\left(\frac{1}{z-2} + \frac{2}{z+4}\right)$$

$$= \frac{1}{3}\left(\frac{z}{z-2} + \frac{2z}{z+4}\right)$$

$\frac{z}{z+4}$: ROC is either $\boxed{|z|<4=|-4|}$, $|z|>4$

$\frac{z}{z-2}$: ROC is either $\boxed{|z|>2}$, $|z|<2$

we want the intersection is $2<|z|<4$.

$$\therefore \left.\begin{array}{l} \dfrac{z}{z+4} \, ; \; |z| < 4 \xrightarrow{\;\mathcal{Z}^{-1}\;} -(-4)^n u[-n-1] \\[2ex] \dfrac{z}{z-2} \, ; \; |z| > 2 \xrightarrow{\;\mathcal{Z}^{-1}\;} 2^n u[n] \end{array} \right\} \leftarrow \text{used basic examples}$$

Thus

$$\frac{1}{z^2+2z-8} \, ; \; 2 < |z| < 4 \xrightarrow{\;\mathcal{Z}^{-1}\;} \frac{1}{3}\left(2^n u[n] + 2 \cdot (-1)(-4)^n u[-n-1] \right)$$

$$= \frac{2^n}{3} u[n] - \frac{2 \cdot (-4)^n}{3} u[-n-1] \, . \quad \blacksquare$$